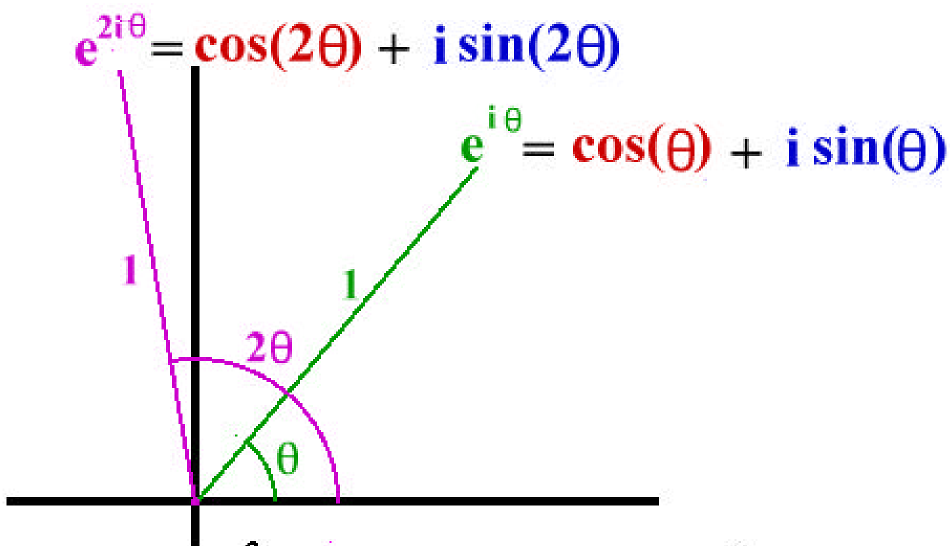
Complex Numbers and Trigonometry
Third Edition

by

Abraham P. Hillman
University of New Mexico

Gerald L. Alexanderson
Santa Clara University

Mervin E. Newton
Thiel College



$$e^{2i\theta} = \left(e^{i\theta}\right)^2 = \left(\cos(\theta) + i\sin(\theta)\right)^2$$

$$= \cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

To

Josephine Hillman

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Preface

This treatment of trigonometry makes it easier to derive and much easier to remember key concepts. It has proved to be most helpful for further study in mathematics, science, and engineering. The original text *Functional Trigonometry* by Hillman and Alexanderson was used successfully in many high schools during the 70's and mid 80's. The predecessors of this short form were very helpful as supplements at The University of New Mexico and at Santa Clara University in various courses, especially for students who were not quite ready for calculus. This version has been augmented with examples and exercises which make use of a graphics calculator. The only difference between this Third Edition and the previous edition (the Revised Edition) is that this one uses the newer HP 49G+, whereas the previous edition used the HP 48GX. The Revised Edition is no longer available, but the original edition is still on the Web. A reader who wishes to see the calculator examples on the HP 48GX will find them in the original edition. Although the calculator chosen here is the HP 49G+, the examples and problems could just as well be done on almost any graphics calculator. On the HP48GII the commands will be exactly as they are on the 49G+. All the examples are given using RPN logic, so the reader using a calculator with algebraic logic will have to make extensive changes to the keystroke sequences given in the examples of this text.

The HP 49G+ has three shift keys, the yellow ALPHA shift, the green left shift and the red right shift. In the examples in this text these shift keys will be abbreviated AS, LS, and RS respectively. The four arrow keys; up arrow, down arrow, left arrow, and right arrow; will be abbreviated UA, DA, LA, and RA respectively. In some cases, especially with the calculator in RPN mode, a shift key must be held down while another key is pressed. In such cases the command will be written as LS(hold). Finally, menu commands will be preceded by the appropriate soft key, F1, F2, ..., F6. To see how this works, let's get our calculator ready for the first calculator example, which is in Section 2 of Chapter 1. The first thing we want to do is to make sure that the calculator is in RPN mode and that the menus will show up as soft keys. To do this press MODE. If the Operating Mode does not show RPN, press the +/- key. Now press F1-Flags and press UA 7 times to highlight system flag 117. If this flag is not checked, press F3-CHK. The flag should now show "Soft MENU." Now press F6-OK twice.

The example for which we are preparing asks us to find a quantity to one decimal place. To accomplish that, we want to set our display mode to Fix 1. Assuming our calculator is now set to the standard display, either of the following sequences will get it ready for Calculator Example 1.2.1:

MODE DA F2-CHOOS DA F6-OK RA 1 F6-OK
LS PRG NXT F4-MODES F1-FMT 1 F2-FIX

For complete instruction on how to set various display modes see page 1-17 of *HP 49G+ Graphing Calculator User's Guide* which came as a PDF file on a CD with the calculator. In the future this publication will be referred to as ***UG***. **NOTE**: ***UG*** can also be found on the Web at http://h10032.www1.hp.com/ctg/Manual/bpia5324.pdf

In the exercises in this text you will at times be asked for exact values and at other times for an approximation to some number of decimal places. In the case of exact answers, a calculator should not be used, in the case of a decimal approximation, a calculator will almost always be needed. Suppose, for example, that you are asked to find the value of c and the value of θ in radians in Figure 1.

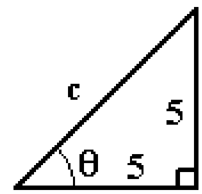


Figure 1

If you are asked for exact values, your responses should be $c = 5\sqrt{2}$ and $\theta = \pi/4$. On the other hand, if c is required to two decimal places and θ to four decimal places, your responses should be $c = 7.07$ and $\theta = 0.7854$ and you would certainly use your calculator to compute these.

Both forms of the solutions are important for different purposes. For theoretical purposes, especially for problem solving by pattern recognition, it is usually much more productive to work with exact values. On the other hand, if c is in inches, and you are to cut a piece of string of that length, the decimal approximation is certainly much more useful.

One final note about calculators. The authors recognize that efficient use of technology is a necessary part of mathematical training, especially for students going into science and engineering. On the other hand, knowing which button to push on the calculator is not a substitute for understanding concepts. In Chapter I, where we are simply reviewing known concepts and terminology, the calculator instructions and examples are integrated throughout the chapter. In Chapters and II and III, however, where the concepts are presumed new to the reader, calculator usage is postponed to the end of the chapters. The reader is encouraged to resist starting the calculator sections until the concepts in the early part of those chapters have been thoroughly mastered. Those readers who prefer not to deal with the calculator can omit all of the calculator examples in Chapter I and the sections on calculator usage at the end of Chapters II and III without loss of essential concepts. It should be noted, however, that many of the calculator examples and exercises do provide additional illustration and clarification of those concepts.

Many of the problems in the exercise sets which follow are required for understanding of concepts in later sections. These problems are distinguished by having their problem numbers in bold face type.

The authors wish to thank, Josephine Hillman, Sara Franco Newton, Dane Haskings (Thiel class of 2006), Kara McDowell (Thiel class of 2001), Amanda McKeehan (Thiel class of 2004), Andrew Murrin (Thiel postgraduate student), Nicole Volchko (Thiel class of 2000), Sean Weaver (Thiel class of 2004), Rebekah Williams (Thiel class of 2004), and Jan Willman (Thiel class of 2004), for their help in proof reading this and the previous editions of this text. Thanks also to Michelle Porada (Thiel class of 2000) and Jonathan Manko (Thiel class of 2004) for their help in preparing this work for the WWW.

The authors would also like to thank Giuseppe Cammarata, who used this text in one of his courses at the Liceo Scientifico Benedetto Croce in Palermo, Italy and Dr. Karl Oman of Thiel College, both of whom made many valuable suggestions for this and/or prefious edition of this text.

An interactive version of this text can be found on the WWW at

http://www.thiel.edu/mathproject/Cnat/

Chapter I

Preliminaries from Geometry

This chapter contains definitions, axioms, and theorems from geometry which are needed for what follows. They serve to insure that the reader and the authors have a common terminology for the material which is prerequisite to the study of trigonometry, so they are, for the most part, presented as facts without formalism or proof.

1. Rays and Segments

A point P on a (straight) line λ divides λ into two half-lines, each of which is a ***ray*** with P as its only endpoint. A ray extends infinitely in one direction. Let P and Q be distinct points and λ be the unique line passing through them. Then the ***ray PQ*** designates the ray with P as endpoint which passes through Q. Also, the ***segment PQ*** consists of P and Q and all the points between these endpoints on the line λ. By specifying P as the initial point and Q as the final point, the segment PQ becomes the "***directed segment***" $\vec{PQ}$.

Directed segments $\vec{AB}$ and $\vec{CD}$ have the same magnitude if the lengths of the segments are equal, as in Figure 1.

Figure 1

If $\vec{AB}$ and $\vec{CD}$ are on parallel lines, (or are on the same line), they may have the same direction, as in Figure 2a, or they may have opposite directions, as in Figure 2b.

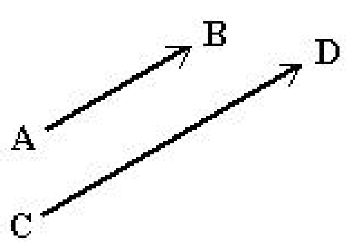


Figure 2a

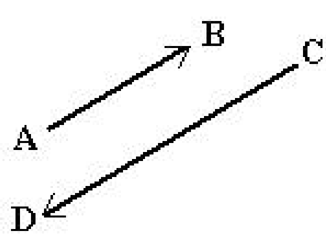


Figure 2b

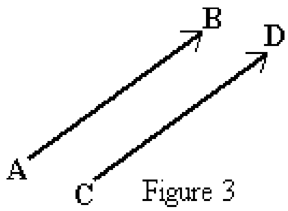


Figure 3

If directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same magnitude and direction as in Figure 3, this will be denoted by the notation

$$\overrightarrow{AB} \cong \overrightarrow{CD}.$$

2. Angles and Triangles

The angle formed by segments *BA* and *BC* (or rays *BA* and *BC*) is denoted by $\angle ABC$; the point *B* is its ***vertex***. If no other angle with vertex at *B* is under consideration, $\angle ABC$ may be shortened to just $\angle B$. Triangle *ABC* is denoted by ΔABC.

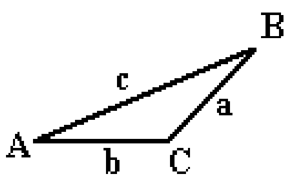


Figure 4a

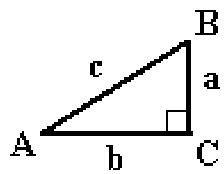


Figure 4b

The sum of the lengths of two sides of a triangle is always greater than the length of the third side. For example, $a + b > c$ in Figures 4a and 4b. Angles are measured either in degrees or radians. The degree measure of a right angle is 90º and its radian measure is $\pi/2$. The sum of the degree measures of the angles of a triangle is 180º and the sum of their radian measures is π. An angle of a triangle is ***acute***, ***right***, or ***obtuse*** depending on whether its degree measure is respectively, less than , equal to, or greater than 90º. The notation $\angle A = 30^{\circ}$ means that the degree measure of $\angle A$ is 30º and $\angle A = \pi/6$ means that the radian measure of $\angle A$ is $\pi/6$.

A triangle with one 90º angle is called a right triangle. The side opposite the right angle is called the ***hypotenuse***. In Figure 4b, $\angle C = 90^{\circ}$ and side *c* is the hypotenuse. The famous Theorem of Pythagoras states that a triangle is a right triangle if and only if the square of the length of one side equals the sum of the squares of the lengths of the other two sides. (For example, $c^2 = a^2 + b^2$ in Figure 4b.) The main steps of a proof follow:

Let ΔABC have a right angle at *C*. Place a square of side *c* externally on the hypothenuse *AB*. Then place copies of ΔABC on the other three sides of the square. All together, we now have a big square whose sides have length $a + b$. See Figure 5.

Now we get

$$c^2 = (a+b)^2 - 4 \cdot \frac{1}{2}ab$$
$$c^2 = a^2 + 2ab + b^2 - 2ab$$
$$c^2 = a^2 + b^2$$

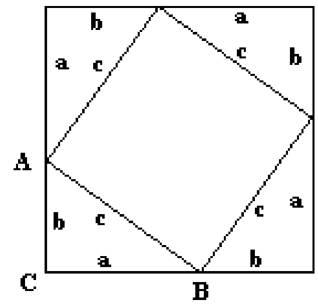


Figure 5

Calculator Example 1.2.1

When laying out the foundation of a new building, stakes are driven at the four corners. Measuring the proper distances between the stakes is relatively easy, but measuring the angles properly is much more difficult. After driving the stakes, the builder always measures the diagonal distance to make sure it satisfies the Theorem of Pythagoras thereby insuring that the angles are 90^o. Suppose that the points *A, B,* and *C* in Figure 4b are three of the four corner stakes of a rectangular foundation, *a* = 36 ft, and *b* = 40 ft. What must *c* be to insure $\angle C = 90^o$? Give your answer to the nearest 10th of an inch.

Solution: We will assume the calculator has already been set to Fix 1 display mode (see Preface). We will let the calculator keep track of the units for us, so the first sequence is to get to the appropriate units menu: RS UNITS F2-LENGTH. For each of the given lengths we will enter the value, attach the units, then square it: 36 F5-ft LS x^2 40 F5-ft LS x^2. We now have the squares of *a* and *b* on the stack. To find *c* we must add these, take the square root, then convert the result to inches: + $\sqrt{x}$ LS F6-in. We see the answer 645.8_in on the display. For complete instructions on the use of units see "Operations with Units" starting on page 3-17 of ***UG***.

3. Similar Triangles

By definition, ΔABC is similar to $\Delta A'B'C'$ if $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$. See Figure 6.

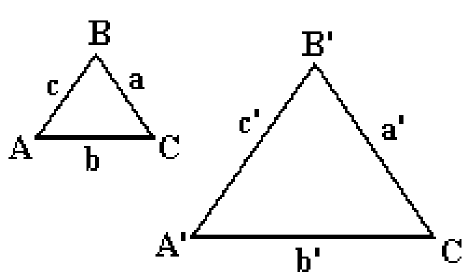


Figure 6

Each of the following conditions guarantees that ΔABC is similar to $\Delta A'B'C'$.

(a) Two angles of one triangle are equal respectively to the corresponding two angles of the other.

(b) Two sides of one triangle are proportional to the corresponding sides of the other triangle and the included angles are equal.

(c) The three sides of one triangle are proportional to the corresponding sides of the other.

If we are given that ΔABC is similar to $\Delta A'B'C'$ then

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C', \text{ and } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

As stated in (a), (b), and (c), to prove that ΔABC is similar to $\Delta A'B'C'$, it suffices to show that $\angle A = \angle A'$ and $\angle B = \angle B'$ or to show that $\frac{a}{a'} = \frac{b}{b'}$ and $\angle C = \angle C'$ or to show that $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Calculator Example 1.3.1

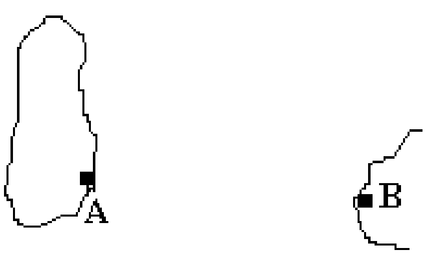


Figure 7a

While sailing around the island in Figure 7a you hit a submerged rock and tore a hole in your boat. You are now stuck on the island and wonder how far it is from point *A* on the island to point *B* on the mainland so you can decide if you can risk swimming it. You have a compass, a tape measure, and your trusty HP 49G+ calculator. How do you estimate the distance from *A* to *B*?

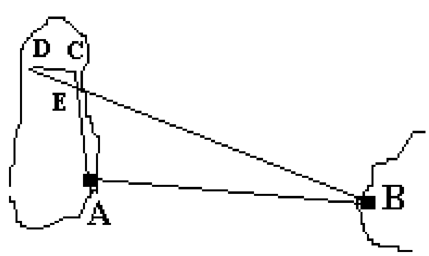


Figure 7b

Solution: From *A* use the compass to measure the bearing to *B*. Now walk along the beach to some point *C* making a line in the sand as you go. See Figure 7b. From *C* walk along the bearing 180º from the bearing you measured from *A* to *B* to some point *D* from which you can still see *B*. Put a marker at *D* and walk strait towards *B* until you get to the line *AC* and mark the point *E*.

Notice that since *AB* is parallel to *CD*, $\angle A = \angle C$. (See Section 5 below.) Also, $\angle AEB = \angle CED$, so ΔAEB is similar to ΔCED. We now have $\frac{AB}{CD} = \frac{AE}{CE}$. Using the tape measure you find that *CE* = 21 yd, *CD* = 67 yd, and *AE* = 113 yd. Now solve the previous proportion for *AB* and substitute the measurements, giving $AB = \frac{113 \cdot 67}{21}$. The sequence 113 ENTER 67 × 21 ÷ on the calculator shows the distance to be a bit over 360 yards.

4. Important Special Triangles

Two sides of a triangle have equal length if and only if the angles opposite them have equal measure. Such a triangle is called an ***isosceles*** triangle. It follows that the angles of an ***equilateral triangle*** (one having all sides equal) each measures 60^o.

Let ΔABC be equilateral with each side having 2 units as its length. Let M be the midpoint of side AC. See Figure 8. Then ΔAMB and ΔCMB are congruent right triangles, $\angle ABM = 30^o$, $\angle A = 60^o$, $\angle AMB = 90^o$, side $AM = 1$, and the length h of side MB satisfies

$$h^2 + 1^2 = 2^2.$$

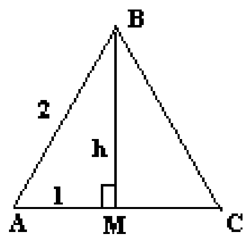


Figure 8

Thus $h = \sqrt{3}$ and the three sides of this 30^o, 60^o, 90^o triangle have lengths 1, $\sqrt{3}$, 2. If ΔEFG is any triangle with $\angle E = 30°$, $\angle F = 60°$, and $\angle G = 90°$, then ΔEFG is similar to ΔBAM and its sides *e, f, g* must be proportional to 1, $\sqrt{3}$, 2. We can write this as $e : f : g = 1 : \sqrt{3} : 2$. It follows that the lengths of the sides of a 30^o, 60^o, 90^o triangle can be written as e, $\sqrt{3}\, e$, $2e$.

Now let ΔABC be an isosceles right triangle with $\angle C = 90^o$, c as the hypotenuse and k as the length of each of the other two sides. Then $\angle A = 45^o = \angle B$ and $c^2 = k^2 + k^2$. It follows that $c = \sqrt{2}k$ and thus the sides of a 45^o, 45^o, 90^o triangle are of the form k, k, $\sqrt{2}k$.

Before computers, when drafting was done by hand, every draftsman had several of these special triangles of various sizes in his tool box. See figure 9a where ΔTUV is a 30^o, 60^o, 90^o triangle and ΔWXY is a 45^o, 45^o, 90^o triangle. With these, the draftsman could construct many angles of various sizes. In figure 9b we see an example of how these two triangles and addition of angles can be used to create a 75^o angle, and in figure 9c we see an example of how one of the triangles and subtraction is used to create an angle of 150^o.

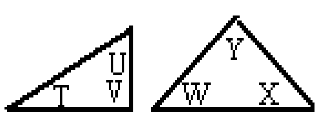


Figure 9a

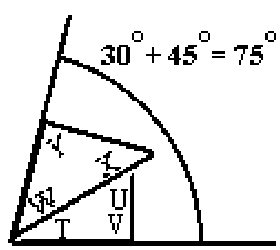


Figure 9b

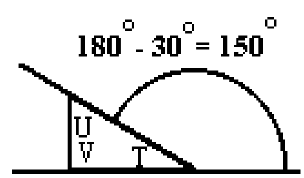


Figure 9c

5. Parallel Lines and Parallelograms

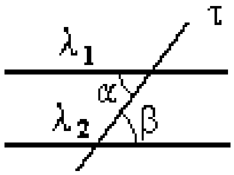


Figure 10

If two lines, λ_1 and λ_2, are cut by a transversal, τ, the lines are parallel if and only if the alternate interior angles α and β are equal. See Figure 10.

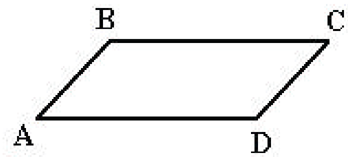


Figure 11

A parallelogram is a quadrilateral whose opposite sides are parallel. It is a theorem that the quadrilateral *ABCD* is a parallelogram if and only if $\vec{AB}$ and $\vec{DC}$ have equal magnitudes and directions, that is, if and only if $\vec{AB} \cong \vec{DC}$. See Figure 11.

Exercises for Chapter I

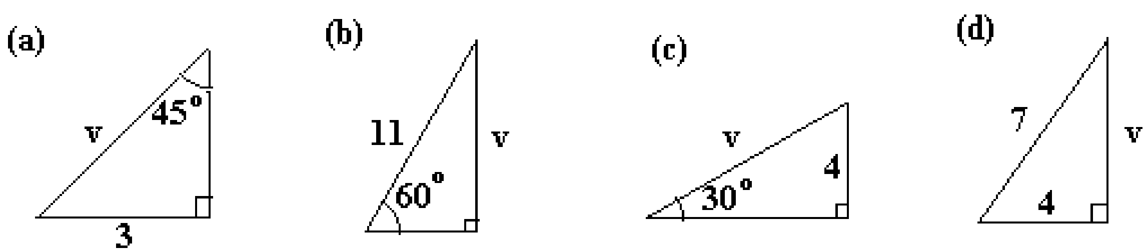
1. Classify each of the following angles as acute, right, obtuse, or not possible in a triangle:

$$-30^o,\ 0^o,\ \sqrt{37}\ ^o,\ 90^o,\ 156^o,\ 180^o,\ 360^o.$$

2. Tell which of the following pairs of angles are possible in the same triangle and find the third angle in each such case:

   (a) $90^o, 90^o$; (b) $100^o, 85^o$; (c) $50^o, 60^o$; (d) $30^o, 140^o$.

3. Find (i) the exact value and (ii) a two decimal approximation (See Preface) for *v* in each of the following triangles:



4. Of the following triples, first identify those which represent sides of a triangle. Of those, select the right triangles and identify them. Then pick out the $45^o, 45^o, 90^o$ triangles and the $30^o, 60^o, 90^o$ triangles:

(a) $\sqrt{5},\ \sqrt{5},\ 10$

(b) 2, 3, 11

(c) 5, 12, 13

(d) 8, 9, 11

(e) 3, 3, $3\sqrt{2}$

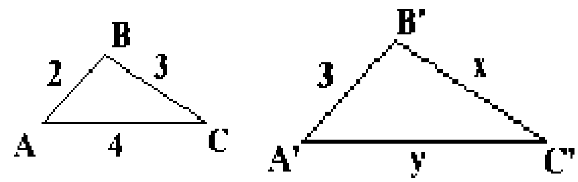
(f) 1, 2, $\sqrt{5}$

(g) 1, $\sqrt{2},\ \sqrt{3}$

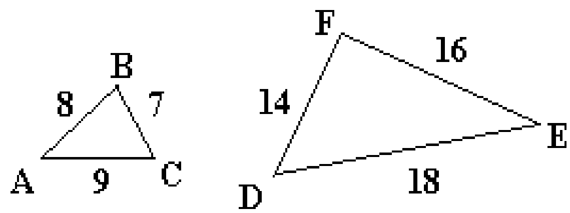
(h) $3\sqrt{3},\ 9,\ 6\sqrt{3}$

(i) 7.11, 9.48, 11.85

5. Given that ΔABC is similar to $\Delta A'B'C'$, find x and y.



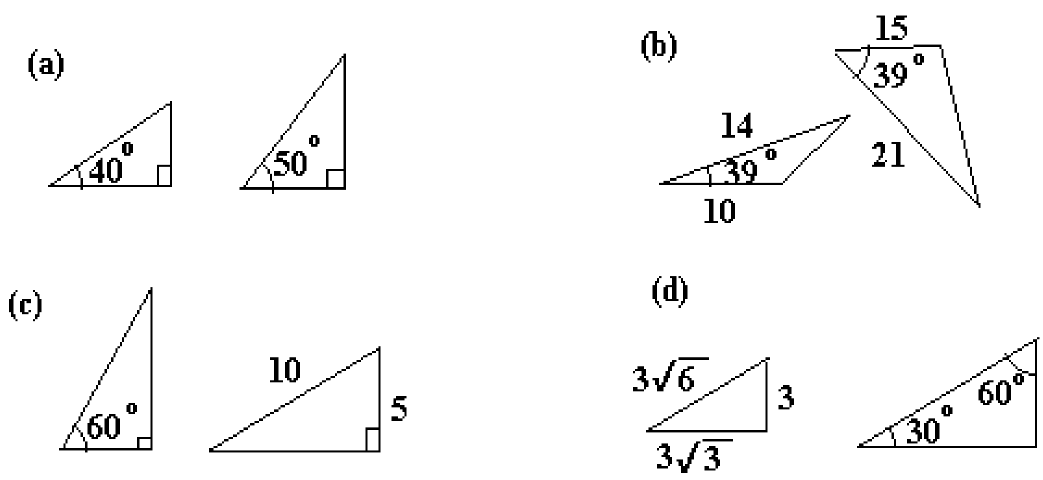
6. Which angle in ΔDEF is equal to angle A?



7 (a) Is a triangle with sides $\sqrt{3},\ 3,\ 2\sqrt{3}$ similar to one with sides $5,\ 10,\ 5\sqrt{3}$?

(b) Is a triangle with sides 23.472, 41.144, 51.256 similar to one with side 55.1592, 96.6884, 110.2004?

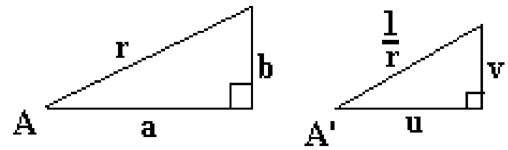
8. Which of the following pairs of triangles are similar? Justify your answers.



9. Given: angles A and A' are equal.

(a) Find u and v in terms of r, a, and b.

(b) If $a = 2.6$ and $b = 1.1$ find r, u, and v to 3 decimal places.



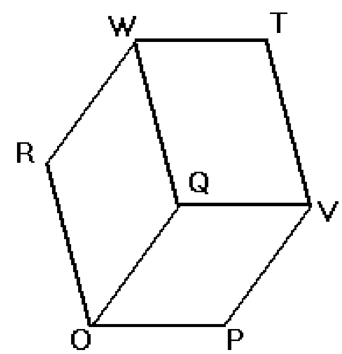
10. Are any two 30º, 60º, 90º triangles similar? Justify your answer.

11. Explain why a triangle whose sides are of length $\sqrt{a}$, $\sqrt{b}$, $\sqrt{a+b}$ is a right triangle. (Of course, $a > 0$ and $b > 0$.)

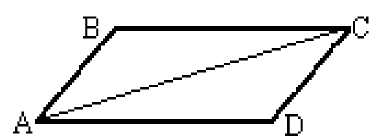
12. Complete the following table for converting certain degree measures to radians or, when read properly, radians to degrees:

θ in degrees	0º	30º	45º			120º		150º	180º
θ in radians	0	π/6		π/3	π/2		3π/4		

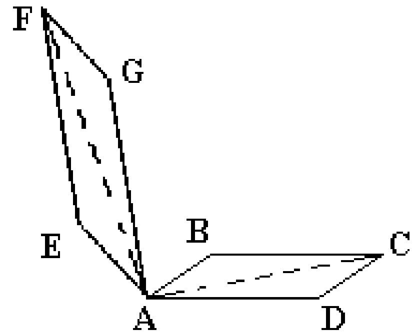
13. Given that $\overrightarrow{OQ} \cong \overrightarrow{PV} \cong \overrightarrow{RW}$ and that $\overrightarrow{VT} \cong \overrightarrow{OR}$, show that $\overrightarrow{WT} \cong \overrightarrow{OP}$.



14. Show that the quadrilateral $ABCD$ is a parallelogram if and only if ΔABC is similar to ΔCDA with $\triangle ACB = \triangle CAD$.



15. Prove that $AEFG$ is a parallelogram, given that $ABCD$ is a parallelogram, $\frac{AE}{AB} = \frac{AF}{AC} = \frac{AG}{AD}$, $\angle DAC = \angle GAF$, and $\angle CAB = \angle FAE$.



16. Use the methods depicted in Figure 9b and in Figure 9c in Section 4 above to show how to construct the angles given below with 30^o, 60^o, 90^o and 45^o, 45^o, 90^o triangles.

(a) 105^o; (b) 15^o; (c) 225^o; (d) 135^o; (e) 195^o.

17. Let $\angle C = \pi/2$ in ΔABC. Let a, b, and c stand for the lengths of sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively. Find:

(a) c when $a = 5$ and $b = 12$

(b) b when $a = 8$ and $c = 17$

(c) $\angle A$ when $b = 4\sqrt{3}$ and $c = 8$

(d) $\angle B$ when $b = 5\sqrt{2}$ and $c = 10$.

18. Given a unit length, outline the construction with straightedge and compass of lengths of

(a) 2/3

(b) $\sqrt{5}$.

Chapter II

Complex Numbers, A Geometric View

1. Polar Form

The complex number system may be regarded as a numerical representation of the points in a plane (called the ***Argand Plane*** in this context). In the Argand Plane, one selects two points and calls them *O* (***origin***) and *U* (***unity***). The distance between *O* and *U* is chosen as the unit length. Then the location of any other point *P* in the plane is specified by polar coordinates $[r,\ \theta]$, where *r* is the distance from *O* to *P* and $\theta = \triangle UOP$. The angle θ is positive when measured counterclockwise and negative when measured clockwise. When θ is measured in radians, we will generally indicate that *P* has $[r,\ \theta]$ as polar coordinates by writing $P = re^{\theta i}$; this borrows a notation from Complex Variables courses. (See Supplementary Problem 6.)

For example, if the distance between *O* and *M* is two units and the angle (measured counterclockwise) from ray *OU* to ray *OM* is $\pi/4$, we write $M = 2e^{(\pi/4)i}$. [See Figure 1a.] Similarly, if the distance between *O* and *N* is ½ and the (clockwise) angle from ray *OU* to ray *ON* is $-2\pi/3$, we write $N = \frac{1}{2}e^{-2\pi i/3}$. [See Figure 1b.] The origin *O* has zero as its *r*-coordinate and any angle may be chosen as its angle; thus $O = 0 \cdot e^{\theta i}$ for all real numbers θ.

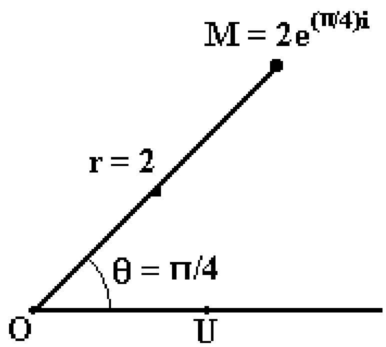


Figure 1a

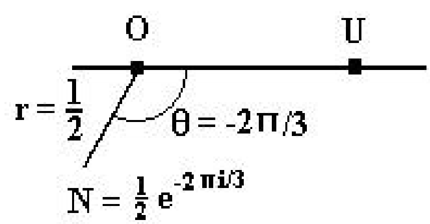


Figure 1b

2. Terminology

The expression $re^{\theta i}$ for a complex number is called its ***polar form***. The nonnegative real number *r* is called the ***absolute value*** or ***modulus*** or ***magnitude*** of *P* and we write $r = |P|$. The angle θ is called an ***argument*** of *P* and is denoted as arg *P*.

A fixed point *P* always has a unique nonnegative real number *r* as its absolute value (i.e., distance to *O*). On the other hand, *P* always has an infinite number of arguments since

$$re^{\theta i} = re^{(\theta \pm 2n\pi)i} \quad \text{for } n = 0,1,2,3,\ldots.$$

For example, some of the other representations for the point $M = 2e^{(\pi/4)i}$ of Figure 1a are $2e^{-7\pi i/4}$, $2e^{9\pi i/4}$, and $2e^{17\pi i/4}$. Also, e^{0i}, $e^{2\pi i}$, $e^{-2\pi i}$, and $e^{-4\pi i}$ are several of the representations of *U*.

The set of points in the Argand Plane is made into the ***Complex Number System*** by defining addition and multiplication as follows:

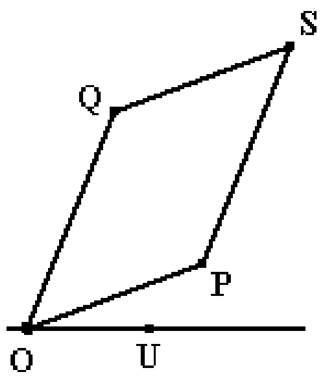


Figure 2

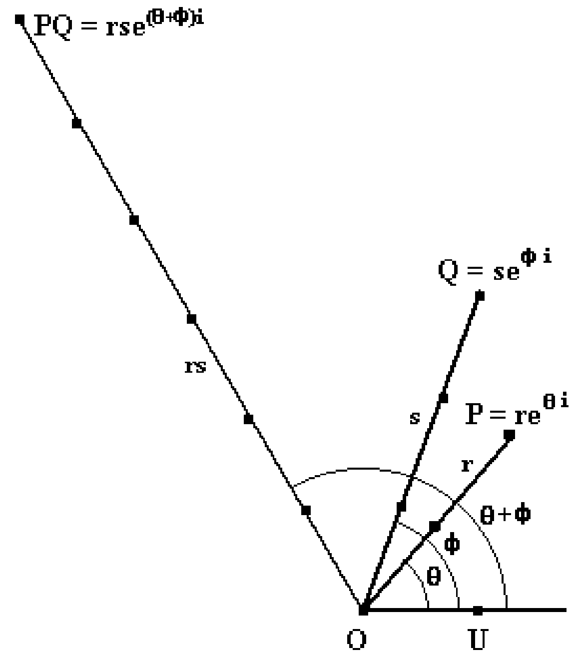


Figure 3

Sum. (Addition of complex numbers) $P + Q$ is the point *S* such that the directed segment $\overrightarrow{PS}$ has the same magnitude and direction as $\overrightarrow{OQ}$. This means that the equation $S = P + Q$ implies that the quadrilateral *OPSQ* is a parallelogram (See Figure 2) unless it collapses into a line segment.

Product. (Multiplication of complex numbers) If $P = re^{\theta i}$ and $Q = se^{\phi i}$, then $PQ = rse^{(\theta+\phi)i}$. Thus *PQ* is the point whose absolute value is the product of the absolute values of *P* and *Q* and whose arguments are the sums of an argument of *P* and an argument of *Q*. This is consistent with the fact that the angles θ and ϕ appear in the exponents. See Figure 3.

Exercises for Chapter II Sections 1 and 2

For all problems calling for a graph, make all diagrams neat and accurate with the unit of length at least half an inch. It may be helpful to use graph paper and a protractor. Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. Let $C = 2e^{-3\pi i/4}$, $D = 4e^{\pi i/6}$, $E = e^{2\pi i}$, and $F = e^{-2\pi i/3}$. Give the polar form (i.e., $re^{\theta i}$ form) for each of the following products.

   (a) CD; (b) CE; (c) CF; (d) C^2.

2. Give an alternate representation for $5e^{(\pi/3)i}$ using a negative argument and one using a positive argument different from $\pi/3$.

3. Let $A = 2e^{(5\pi/4)i}$ and $B = 3e^{(-\pi/3)i}$.

   (a) Give the polar form for AB.

   (b) Plot A, B, and AB.

   (c) Construct $S = A + B$ given that $OASB$ is a parallelogram.

4. Do as in problem 3 with $A = \sqrt{2}e^{(\pi/4)i}$ and $B = 2e^{(2\pi/3)i}$.

5. Let $G = 3e^{0i}$ and $H = 4e^{(\pi/2)i}$. Plot G, H, and $G + H$ and find $|G + H|$, i.e., the distance form O to $G + H$.

6. Let $K = 12e^{(\pi/2)i}$ and $L = 5e^{\pi i}$. Plot K, L, and $K + L$ and find $|K + L|$.

7. Let $M = 4e^{(7\pi/6)i}$ and $N = 2e^{(-\pi/6)i}$. Plot M, N, and $M + N$.

8. Let $M = 2\sqrt{3}e^{(\pi/4)i}$ and $N = 2e^{(3\pi/4)i}$. Plot M, N, and $M + N$.

9. The points $re^{\theta i}$ for which r is any nonnegative real number and θ is in $\{0, 2\pi, -2\pi, 4\pi, -4\pi, \ldots\}$ form the ray OU. Give similar geometric characterizations for the following sets of points:

   (a) The set R of all the points $re^{\theta i}$ with r any nonnegative real number and θ in $\{0, \pi, -\pi, 2\pi, -2\pi, 3\pi, \ldots\}$.

   (b) The set I of all the points $se^{\phi i}$ with s any nonnegative real number and ϕ in $\{\pi/2, -\pi/2, 3\pi/2, -3\pi/2, 5\pi/2, \ldots\}$.

10. Can every point P of the Argand Plane be expressed as $P = A + B$ with A in the set R of Problem 9 (a) and B in the set I of Problem 9 (b)? Explain.

11. Let $P = 6e^{(4\pi/9)i}$. Find the polar form of a point N such that $P + N = O$. Show P,O, and N in a diagram.

12. Let P be as in Problem 11. Find the polar form of a point M such that $PM = U$. Show P, U, and M in a diagram.

13. Let O be the origin and U be the unity in the Argand Plane and let P, Q, and R be any complex numbers. Verify that the complex number system has each of the following properties:

(a) Commutativity of addition: $P + Q = Q + P$.

(b) Associativity of addition: $(P + Q) + R = P + (Q + R)$. HINT: See Problem 13 of Chapter 1.

(c) Additive identity: $O + P = P + O = P$.

(d) Additive inverse: For each P there exists a complex number N such that $N + P = P + N = O$.

(e) Commutativity of multiplication: $PQ = QP$.

(f) Associativity of multiplication: $(PQ)R = P(QR)$.

(g) Zero multiplication: $OP = PO = O$.

(h) Multiplicative identity: $UP = PU = P$.

(i) Multiplicative inverse: For each $P \neq O$ there exists a complex number M such that $MP = PM = U$.

(j) Distributive law: $P(Q + R) = PQ + PR$. HINT: See Problem 15 of Chapter 1.

14. Let P, Q, S, T be four complex numbers. Use the results of Problem 13 to show that:

(a) $(P + Q)^2 = P^2 + 2PQ + Q^2$.

(b) $(S + T)(S - T) = S^2 - Q^2$.

(c) $(P + Q)(S + T) = PS + QS + PT + QT$.

15. Let $E = 4e^{0i}$, $F = 4e^{\pi i}$, $G = 7e^{0i}$, and $H = 7e^{\pi i}$. Find the polar form of:

(a) EG; (b) FH; (c) FG; (d) EH.

16. Let $E, F, G,$ and H be as in Problem 15. Find the polar form of:

(a) $E + G$; (b) $F + H$; (c) $F + G$; (d) $E + H$.

17. Let P and Q both be in the set R of Problem 9 (a).

(a) Is the product PQ also in the set R? Explain.

(b) Is the sum $P + Q$ also in R? Explain.

(c) Is $Qe^{(\pi/2)i}$ in the set I of Problem 9 (b)? Explain.

18. Let $Q = 4e^{(5\pi/3)i}$ and $\overline{Q} = 4e^{(-5\pi/3)i}$. Show Q, $\overline{Q}$, $Q\overline{Q}$, and $Q + \overline{Q}$ in a diagram.

19. Let $A = 1024e^{(\pi/4)i}$ and $C = e^{(2\pi/5)i}$.

(a) Verify that $C^5 = \left(C^2\right)^5 = \left(C^3\right)^5 = \left(C^4\right)^5 = U$.

(b) Find in polar form a complex number B such that $B^5 = A$.

(c) Find in polar form and plot B, CB, C^2B, C^3B, C^4B, and C^5B.

(d) Verify that $(CB)^5 = \left(C^2B\right)^5 = \left(C^3B\right)^5 = \left(C^4B\right)^5 = A$.

20. Find in polar form and plot 5 fifth roots of $243e^{(-13\pi/18)i}$. What kind of geometrical figure has these 5 points as vertices?

21. Find in polar form and plot 7 seventh roots of $128e^{(14\pi/19)i}$.

22. Let n be an integer greater than 1 and let $D = re^{\theta i}$ be any complex number. You may assume $0 \le \theta < 2\pi$.

(a) Find the complex number F with the smallest possible positive argument such that $F^n = U$.

(b) Find a complex number E such that $E^n = D$.

(c) Verify that the complex numbers EF^k, for $k = 0, 1, 2, ..., n - 1$ are distinct nth roots of D.

3. Negative of a Point, Subtraction, Conjugate

As in other number systems, if $P + N = O$, one writes $N = -P$ and calls N the ***negative*** of P. It follows from the definition of addition of points in the Argand Plane that N is the point such that the directed line segment $\overrightarrow{ON}$ has the same magnitude and direction as $\overrightarrow{PO}$; i.e., N is the point such that O is the midpoint of segment PN. If $P = re^{\theta i}$, then clearly

$$N = re^{(\theta+\pi)i} = re^{(\theta-\pi)i}.$$

See Figure 4.

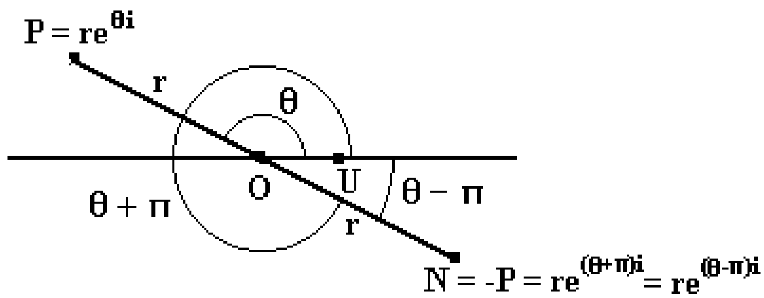


Figure 4

The ***difference*** $E - F$ is the point G such that $E = F + G$. One can also obtain the difference $G = E - F$ using the formula $G = E + (-F)$. See Figure 5.

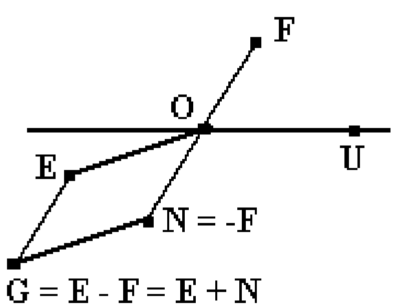


Figure 5

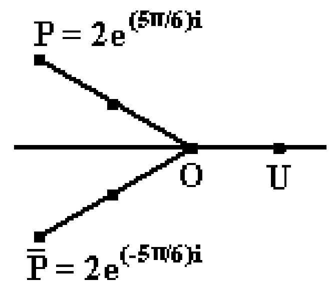


Figure 6

In the Argand Plane, the ***conjugate*** of a point $P = re^{\theta i}$ is the point $\overline{P} = re^{-\theta i}$, with the same absolute value as P but with the argument the negative of that of P. Note that a point P and its conjugate $\overline{P}$ are symmetrically situated with respect to the straight line determined by O and U. See Figure 6.

4. Reciprocal of a Point, Division

If $Q \neq O$ and $MQ = U$, one writes $M = Q^{-1}$ and this is called the ***reciprocal*** of Q. It is clear from the definition of multiplication that if $Q = re^{\theta i}$ with r not equal to zero, then

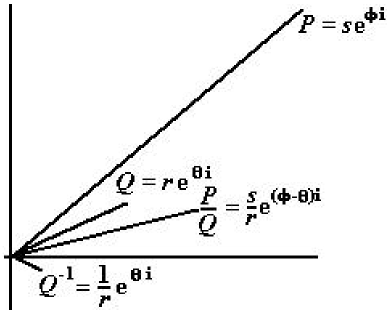


Figure 7

$Q^{-1} = \frac{1}{r}e^{-\theta i}$. The division of a complex number P by a complex number Q which is not O can now be defined by $P \div Q = \frac{P}{Q} = PQ^{-1}$. Thus, if $P = se^{\phi i}$, then $\frac{P}{Q} = PQ^{-1} = \left(se^{\phi i}\right)\left(\frac{1}{r}e^{-\theta i}\right) = \frac{s}{r}e^{(\phi-\theta)i}$. See Figure 7.

For example, if $P = 6e^{(2/9)\pi i}$ and $Q = 2e^{(5/36)\pi i}$ then

$$\frac{P}{Q} = \frac{6}{2}e^{\left(\frac{2}{9}-\frac{5}{36}\right)\pi i} = 3e^{(\pi/12)i}.$$

5. The Real and Imaginary Axes; Rectangular Form

Each point on the straight line through the origin O and the unity point U is expressible as $re^{\theta i}$ with $\theta = 0$ or π. The set of all these points is closed under addition and multiplication. (See Problem 17 in Exercises for Chapter 2 Sections 1 and 2) In fact, these points behave just like the real numbers under addition, subtraction, multiplication, and division. For this reason, the line through O and U is called the ***real axis*** and we identify each real number with a point on the real axis in the following manner.

The real number zero is identified with the origin O. A positive (real) number r is identified with the point re^{0i}. The material in Section 3 of this chapter makes it natural for its negative $-r$ to represent the point $re^{\pi i}$. In particular, we have $U = e^{0i} = 1$.

The line perpendicular to the real axis at the origin is called the ***imaginary axis***. The imaginary unit point $e^{(\pi/2)i}$ is designated as i. Then the points $re^{(\pi/2)i}$ may be written as ri and points $re^{(3\pi/2)i}$ as $-r$i. An important fact is that $i^2 = e^{\pi i/2}e^{\pi i/2} = e^{\pi i} = -1$; that is, $i^2 = -1$.

Now every point on the real axis is represented by a real number a and every point on the imaginary axis by a ***pure imaginary number***, i.e., a number bi with b real. Note that a and b may be positive, zero, or negative. See Figure 8 below.

Let P be any point in the Argand Plane. Then the foot of the perpendicular from P to the real axis has a representation as some real number a. Similarly, the foot of the perpendicular from P to the imaginary axis is a pure imaginary number bi. The rule for adding points shows that

$P = a + b\text{i}$. [The parallelogram with vertices 0, a, P, bi turns out to be a rectangle in this case. See Figure 9.]

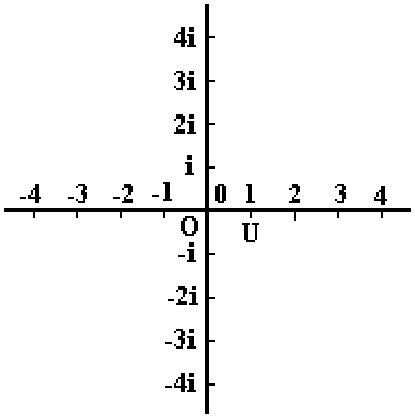


Figure 8

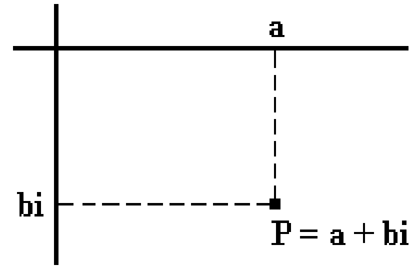


Figure 9

The representation $a + b\text{i}$, with a and b real, is called the ***rectangular form*** of a complex number. The real number a is called the ***real part*** and the real number b is called the ***imaginary part*** of the complex number.

It can be shown (see Problem 17 in the exercises for this section) that in rectangular form the complex numbers have the following rules:

ADDITION $\quad (a + b\text{i}) + (c + d\text{i}) = (a + c) + (b + d)\text{i}.$

SUBTRACTION $\quad (a + b\text{i}) - (c + d\text{i}) = (a - c) + (b - d)\text{i}.$

MULTIPLICATION $\quad (a + b\text{i})(c + d\text{i}) = (ac - bd) + (ad + bc)\text{i}.$

Example 1. Convert the polar form $P = 5\sqrt{2}\,e^{(3\pi/4)\text{i}}$ to rectangular form $a + b\text{i}$.

Solution: Let H and V be the feet of the perpendiculars from P to the real axis and the imaginary axis, respectively. We see that ΔPHO is a 45°, 45°, 90° triangle. Hence the lengths of its sides are in the ratio $1{:}1{:}\sqrt{2}$. Since the hypotenuse has length $5\sqrt{2}$, the two equal sides must have length 5. Then $H = 5e^{\pi\text{i}} = -5$ and $V = 5e^{(\pi/2)\text{i}} = 5\text{i}$. Hence $P = H + V = -5 + 5\text{i}$. See Figure 10.

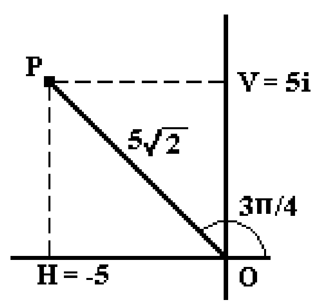


Figure 10

Example 2. Convert the rectangular form $Q = -7 - 7\sqrt{3}i$ to polar form.

Solution: Let C = -7 and $D = -7\sqrt{3}i$. [See Figure 11.] Clearly ΔQDO is a right triangle. Since the ratio of the length of side *DO* to the length of the side *QD* is $\sqrt{3}$, it is a 30°, 60°, 90° triangle and the hypotenuse *OQ* has twice the length of the shortest side *QD*, that is, the hypotenuse has length 14. Also, the counterclockwise angle from ray *OU* to ray *OQ* is 240° i.e., $4\pi/3$. Hence $Q = 14e^{(4\pi/3)i}$.

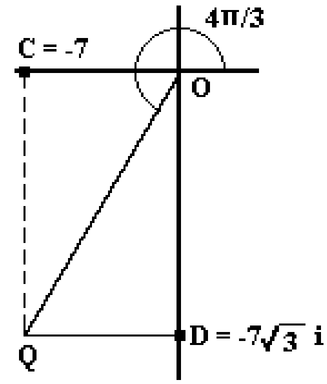


Figure 11

Example 3. Find the rectangular form of $P = 2\sqrt{2}e^{(5\pi/12)i}$.

Solution: First we note that $5\pi/12$ radians equals 75°, or 30° plus 45°. Using a 30°, 60°, 90° triangle and a 45°, 45°, 90° triangle [see Figure 12], one finds that

$$\sqrt{2}e^{(\pi/4)i} = 1 + i,$$

$$2e^{(\pi/6)i} = \sqrt{3} + i.$$

Multiplying these two complex numbers, one has

$$\sqrt{2}e^{(\pi/4)i}2e^{(\pi/6)i} = (1 + i)(\sqrt{3} + i)$$

$$2\sqrt{2}e^{(5\pi/12)i} = (\sqrt{3} - 1) + (\sqrt{3} + 1)i.$$

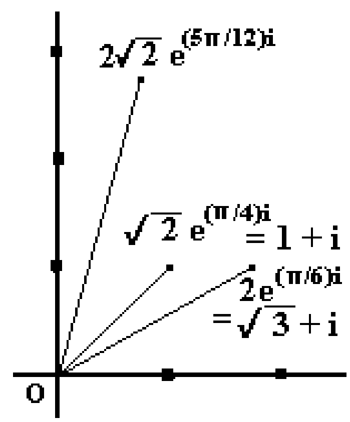


Figure 12

Exercises for Chapter II Sections 3, 4, and 5

Do NOT use a calculator for any of these problems, Give only exact answers. (See Preface)

1. Convert the following polar forms to rectangular form *a* + *b*i.

   (a) $8e^{(\pi/6)i}$; (b) $8e^{(5\pi/6)i}$; (c) $\sqrt{2}e^{(5\pi/4)i}$; (d) $9e^{(3\pi/2)i}$; (e) $4e^{(-\pi/4)i}$.

2. Convert the following rectangular forms to polar form $re^{\theta i}$. Give angles in radians.

   (a) 1 + i; (b) 7i; (c) $-5\sqrt{3} + 5i$; (d) -3; (e) -4 - 4i.

3. Find the rectangular form $a + bi$ of $e^{\theta i}$ for

$$\theta = 0,\ \pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6,\ \text{and}\ \pi.$$

4. Do as in Problem 3 for $\theta = 7\pi/6,\ 5\pi/4,\ 4\pi/3,\ 3\pi/2,\ 5\pi/3,\ 7\pi/4,\ 11\pi/6,\ \text{and}\ 2\pi$.

5. Find the rectangular form for the conjugate $\overline{P}$ of $P = -7 + 8i$.

6. Find the rectangular form for the conjugate $\overline{P} = re^{-\theta i}$ of $P = re^{\theta i} = a + bi$, where a and b are real numbers.

7. Let $S = 13e^{\phi i} = -12 + 5i$. Find both the polar and the rectangular form of the point symmetric to S with respect to:

(a) the real axis; (b) the imaginary axis; (c) the origin;

(d) the straight line through O and $1 + i$.

8. Do the same as in Problem 7 for $T = re^{\theta i} = h + ki$ instead of S.

9. Let $A = 5e^{\alpha i} = 3 + 4i$ and $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6i$. Find in both polar and rectangular form:

(a) $-A$; (b) $-B$; (c) $\overline{A}$; (d) $\overline{B}$; (e) AB; (f) $A\overline{B}$.

10. For A and B of Problem 9, give the rectangular form of $A - B$ and plot O, A, $A - B$, and $-B$. What kind of quadrilateral are these four points the vertices of?

11. Let $A = 5e^{\alpha i} = 3 + 4i$ and $D = 13e^{\beta i} = 12 - 5i$. Find the rectangular form of :

(a) $65e^{(\alpha+\beta)i}$; (b) $65e^{(\alpha-\beta)i}$.

12. Let $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6i$ and $C = 2e^{(\pi/6)i} = \sqrt{3} + i$. Find $12\sqrt{2}e^{(7\pi/12)i}$ in rectangular form.

13. Given that $re^{\theta i} = 7 + 5i$, use the Pythagorean Theorem to find r.

14. Given that $8e^{\phi i} = 5 + bi$ and $b < 0$, find b.

15. Explain why $|a + bi| = \sqrt{a^2 + b^2}$ for all real a and b and find $|11 - 8i|$.

16. Use the results of Problem 16 in Chapter 1 to find complex numbers in polar and rectangular form with magnitude 1 and arguments equal to those listed below but converted to radians.

(a) 105°; (b) 15°; (c) 225°; (d) 135°; (e) 195°.

17. Verify the rules for addition, subtraction, and multiplication of complex numbers given on page 17.

18. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\phi i} = c + di$. Show that :

(a) $\overline{P + Q} = \overline{P} + \overline{Q}$; (b) $\overline{PQ} = \overline{P} \cdot \overline{Q}$.

19. Let $A = 5e^{\alpha i} = 3 - 4i$. Find in both polar and rectangular form:

(a) iA; (b) $-A$; (c) $-iA$; (d) $\overline{A}$; (e) $i\overline{A}$; (f) A^2; (g) A^3.

20. Let $B = 29e^{\beta i} = -21 + 20i$. Find the polar form (in terms of β) for :

(a) -20 - 21i; (b) 21 - 20i; (c) 20 + 21i; (d) -21 - 20i; (e) 20 - 21i.

21. Let $P = 5 - 5\sqrt{3}i$. Find P, $\overline{P}$, $-P$, and $-\overline{P}$ in both polar and rectangular form.

22. Let $Q = 2\sqrt{2}e^{(3\pi/4)i}$. Find Q, $\overline{Q}$, $-Q$, and $-\overline{Q}$ in both polar and rectangular form.

23. Given that $P = 13e^{\theta i} = 12 - 5i$ and $Q = 5e^{\phi i} = 4 + 3i$, find in rectangular form:

(a) $\overline{P} = 13e^{-\theta i}$; (b) $Q\overline{P} = 65e^{(\phi - \theta)i}$; (c) $-\overline{P} = 13e^{(\pi - \theta)i}$;

(d) $Q^2 = 25e^{2\phi i}$; (e) $PQ = 65e^{(\theta + \phi)i}$.

24. Let $A = 25e^{\alpha i} = 7 + 24i$ and $B = \sqrt{13}e^{\beta i} = 2 - 3i$. For each of the following angles θ, find r, a, and b such that $re^{\theta i} = a + bi$:

(a) $\theta = \alpha - \beta$; (b) $\theta = 2\beta$; (c) $\theta = \alpha - \pi$; (d) $\theta = \alpha + (3\pi/2)$.

25. Express $(\sqrt{3} + i)^{10}(2 - 2\sqrt{3}i)^9$ in $a + bi$ form by first converting $\sqrt{3} + i$ and $2 - 2\sqrt{3}i$ to polar form, raising to powers and multiplying in polar form, and finally converting back.

26. Do as in Problem 25 for $(5 + 5i)^{11}(-7i)^8$.

27. Let $P = 13e^{\theta i} = 5 + 12i$, $Q = 13$, and $S = P + Q$.

(a) Explain why O, Q, S, P are the vertices of a rhombus (i.e., a parallelogram with all four sides equal).

(b) Explain why $\triangle QOS = \frac{1}{2}\triangle QOP$.

(c) Find the absolute value s of S and then find $R = \left(\frac{\sqrt{13}}{s}\right)S$ in $a + bi$ form.

(d) Let R be as in (c). Explain why R and $-R$ are the 2 square roots of P.

28. Let $P = re^{\theta i} = a + bi$, $Q = r$, $S = P + Q$, and $s = |S|$. Explain why each of the following is true.

(a) O, Q, S, and P are the vertices of a rhombus.

(b) $\triangle QOS = \frac{1}{2}\triangle QOP$.

(c) $r^2 = a^2 + b^2$, $s^2 = 2r^2 + 2ra$, and $s = \sqrt{r}\sqrt{2r + 2a}$.

(d) The square roots of P are $\pm\left(\frac{\sqrt{r}}{s}\right)S = \pm\frac{S}{\sqrt{2r + 2a}} = \pm\frac{P + r}{\sqrt{2r + 2a}}$.

29. Let $P = re^{\theta i} = a + bi$ with $b > 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} + i\sqrt{\frac{r - a}{2}}\right)$.

30. Let $P = re^{\theta i} = a + bi$ with $b < 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} - i\sqrt{\frac{r - a}{2}}\right)$.

31. Let $V = e^{\theta i} = c + si$. Show that the square roots $e^{\theta i/2}$ and $e^{(\theta+2\pi)i/2}$ of V are

$$\pm\sqrt{\frac{1+c}{2}} \pm i\sqrt{\frac{1-c}{2}}$$

where like signs are used if $s > 0$ and unlike signs if $s < 0$.

32. Use Problem 31 to find $e^{(\pi/8)i}$ in $a + bi$ form.

33. Let $D = B - A$. See figure 13.

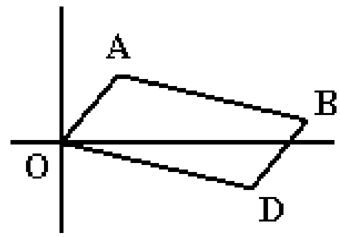


Figure 13

(a) Explain why $\vec{OD}$ and $\vec{AB}$ have the same magnitude and hence $|B - A|$ equals the distance between A and B.

(b) If $A = u + vi$ and $B = x + yi$, show that

$$|B - A| = \sqrt{(x-u)^2 + (y-v)^2}.$$

34. Let $A = 5 + 2i$, $B = 9 + 5i$, and $C = 5 + 5i$.

(a) Explain why $\angle ACB$ is a right angle.

(b) Find sides AC and CB of ΔABC and then use the Pythagorean Theorem to find side AB (i.e., the distance between A and B.)

(c) Find $|B - A|$ and $|B| - |A|$ and tell which equals the distance between A and B.

35. Sketch and identify the locus of all points A in the Argand Plane such that $|A - i| = 3$. (That is, give the graph in the Argand Plane of the equation $|A - i| = 3$.)

36. Do as in Problem 35 for each of the following equations.

(a) $|A - 4| + |A - 3i| = 5$.

(b) $|A - 4| - |A - 3i| = 0$.

37. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\theta i} = c + di$ be two complex numbers with the same argument θ, $\theta \neq 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \cdots$. Also let $r > 0$ and $s > 0$. Explain why each of the following is true:

$$\frac{a}{r} = \frac{c}{s}, \qquad \frac{b}{r} = \frac{d}{s}, \qquad \frac{b}{a} = \frac{d}{c}.$$

38. Perform each of the following divisions in polar form.

(a) $12e^{(7/4)\pi i} \div 4e^{(5/8)\pi i}$; (b) $24e^{(\pi/4)i} \div 6e^{(\pi/3)i}$; (c) $15e^{(3/4)\pi i} \div 12e^{(-\pi/3)i}$.

39. Let P and Q be complex numbers with $|Q| = s > 0$. Show that $\frac{P}{Q} = \frac{1}{s^2} P\overline{Q}$.

40. Use the result of Problem 39 to perform each of the following divisions.

(a) $\frac{10+11i}{3+2i}$; (b) $\frac{9-38i}{4-3i}$; (c) $\frac{11+3i}{2+2i}$; (d) $\frac{29+31i}{2-7i}$.

41. Show that if $c + di \neq 0$, then $\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$.

42. Let $A = 6$ and $B = 4 + 3i$.

(a) Plot the points A and B and complete the ΔOAB.

(b) Find the rectangular form of $A' = Ae^{(2/3)\pi i}$ and $B' = Be^{(2/3)\pi i}$.

(c) On the same graph you used for part (a), plot the points A' and B' and complete the $\Delta OA'B'$. How would you describe the effect that multiplying by $e^{(2/3)\pi i}$ had on the ΔOAB?

43. Let $A = 5 - i$ and $B = 5 + i$. Rotate ΔOAB 30° counterclockwise about the origin. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

44. Let $A = 2 + 5i$ and $B = -1 + 4i$. Rotate ΔOAB 135° clockwise about the origin. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

45. Let $A = -1 - 2i$ and $B = 3 - 2i$. Rotate ΔOAB 60° counterclockwise about the origin and at the same time stretch it so that each side is twice the length of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

46. Let $A = 6 + 12i$ and $B = -3 + 15i$. Rotate ΔOAB 120° clockwise about the origin and at

the same time shrink it so that each side is one third the length of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

47. Let $A = 5 - 2i$ and $B = 3 + 2i$. Rotate ΔOAB 45° counterclockwise about the origin and at the same time stretch it so that the area is twice the area of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

48. Let $A = 4 + 3i$ and $B = 2 + 5i$. Rotate ΔOAB counterclockwise and stretch it until vertex A is at $-5 + 15i$. Find the new coordinate of B and graph the triangle before and after the rotation.

6. Complex Numbers on the Calculator - Polar Form

Our first job is to get the calculator options set up for the task at hand. Key MODE DA DA and F2-CHOOS to see a drop down list with three choices of angle measure. Use DA or UA as needed to select Degrees from the list then press F6-OK. This puts the calculator in degree mode. Now key DA and F2-CHOOS to see three choices for coordinate systems. Again use DA or UA as needed to select Polar from this list then press F6-OK. For more information about angle modes and coordinate systems see page 1-22 of ***UG***. To make sure that the constants in the calculator give numeric rather than symbolic results. Key F1-FLAGS then use DA and F3-CHK as needed to make sure that Flags 2 and 3 are checked and Flag 27 is clear, then F6-OK. For more information about the roll of Flag 2 see page 2-64 of ***UG***. Finally press F3-CAS DA DA to highligt "_Approx." The other two items on that row are "_Numeric" on the left and "_Complex" on the right. Use LA, RA, and F3-CHK as needed to check all three of these items, then press F6-OK on this dialog box and on the next to return to the main screen.

Now look at the row of annunciators at the top of the screen. They should be "Degree," "R∠Z," "HEX," "C~." and "'X'."

In the calculator, the complex number $P = re^{\theta i}$ is expressed as $(r, \measuredangle\theta)$. One limitation of the calculator is that it will not accept variables for the magnitude and argument, only numbers. In particular, if $P = 3e^{(\pi/4)i}$, it can't be expressed as $(3, \measuredangle\frac{\pi}{4})$, it must be expressed as $(3, \measuredangle.785398163398)$. As a general rule, this limitation makes working with degrees much easier than with radians. We will, therefore, start with degrees, but we will see some tricks that will make working with radians not quite as bad as it looks. Actually, working in degrees has its own much more subtle problems. Since the calculator actually works with radians internally, there are two conversions in moving data from the keyboard to the display, and these sometimes cause annoying roundoff errors. To avoid this, set your display to Fix 2 for the time being.

NOTE: The angle mark, $\measuredangle$, is created with the key stroke sequence AS RS 6.

Calculator Example 2.6.1

Let $P = (2,\angle 43^o)$ and $Q = (3,\angle 29^o)$, find PQ. With the calculator in degrees and polar coordinates, key in LS ( ) 2 (NOTE: At this point you may key in the comma which takes two key strokes or a space, SPC, and the calculator will convert it to a comma after you hit ENTER, but neither is necessary.) AS RS 6 43 ENTER LS ( ) 3 AS RS 6 29 ×. You should see $(6.00,\angle 72.00)$ on the display. This clearly satisfies the definition of product from Section 2 of this chapter.

Now try it with $P = (3.4,\angle 87^o)$ and $Q = (6.3,\angle 112^o)$. The result you get is $(21.42,\angle -161.00)$. The magnitude is certainly correct, but we were expecting an argument of 199. What happens is that the calculator always normalizes the argument so that $-180^o < \theta^o \le 180^o$, or in radians, $-\pi < \theta \le \pi$.

Calculator Example 2.6.2

Let $P = (4,\angle 30^o)$ and $Q = (2,\angle 150^o)$ and find $P + Q$. Key in the complex numbers as in the previous example, and press + to find the sum. The result is $(3.46,\angle 60.00)$ to two decimal places. We now use the definition of sum in Section 2 and geometry to verify that this answer is correct. See Figure 14.

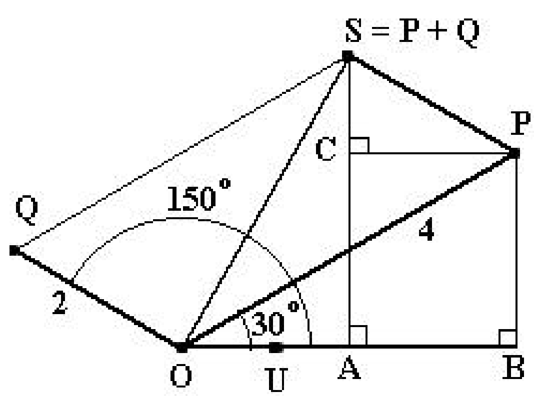


Figure 14

Construct *SA* and *PB* perpendicular to *OU*, and *PC* parallel to *OU*. Now ΔOBP is a 30º, 60º, 90º triangle with hypotenuse 4, so *PB* = 2 = *CA*, and $OB = 2\sqrt{3}$. Since $\angle CPS = 180^o - 150^o = 30^o$, ΔCPS is a 30º, 60º, 90º triangle with hypotenuse 2, so $CP = \sqrt{3} = AB$ and *SC* = 1. *OA* = *OB* - *AB* = $\sqrt{3}$ and *AS* = *AC* + *SC* = 3, so ΔOAS is a 30º, 60º, 90º triangle. Thus $S = (2\sqrt{3},\angle 60^o)$, which agrees with our calculator answer at least to the level of accuracy possible on the calculator.

Calculator Example 2.6.3

An engineer wishes to build a straight railroad from town *A* to town *B* with a tunnel through the mountain from point *C* to point *D*. [See Figure 15] She cannot, of course see *B* from *A*, so she does not know what direction to go from *A* to find one end of the tunnel at *C*, nor what direction to go from *B* to find the other end of the tunnel at *D*. She can, however, see a tall tree at *P* from *A*, from *P* she can see a large rock at *Q*, and from *Q* she can see *B*. How should she proceed?

Solution: She visualizes the map as the Argand Plane with the origin at *A* and *U* one kilometer east of *A*. She finds that from *A* to *P* is 10.13 kilometers in a direction 19.8º measured counterclockwise from east, hence $\overrightarrow{AP}$ can be thought of as the complex number $(10.13,\angle 19.8)$.

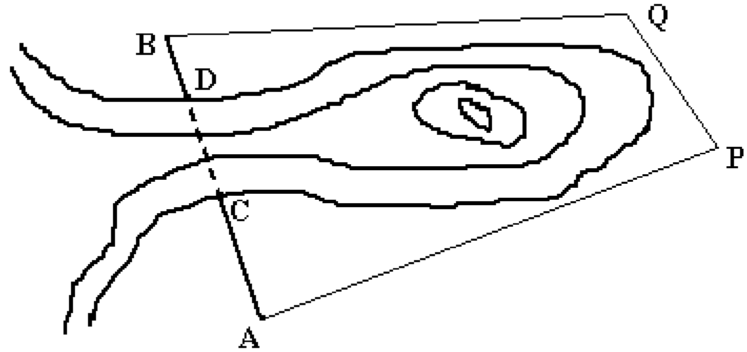


Figure 15

From *P* to *Q* is 125.3º measured counterclockwise from east and the distance is 3.17 kilometers, so $\overrightarrow{PQ}$ can be thought of as the complex number $(3.17,\angle 125.3)$. Finally, she finds that from *Q* to *B* is 8.68 kilometers and the direction is 177.4º clockwise from east, so $\overrightarrow{QB}$ is considered to be the complex number $(8.68,\angle -177.4)$. She adds these three complex numbers on her calculator and finds the sum to be $(5.71,\angle 99.8)$, so from *A* to *B* is a distance of 5.71 kilometers in the direction 99.8º counterclockwise from east. She now goes in that direction from *A* until she comes to the mountain at a point she labels *C*, then goes in the opposite direction from *B* until she comes to the mountain at the point she labels *D*.

As mentioned above, the calculator's representation of a complex number $P = (r,\angle\theta)$ requires *r* and θ to be expressed in decimal form. If $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$ one can compute the magnitude and argument to twelve significant figures on the calculator, then key them into the parentheses to get $(3.46410161514,\angle .448798950513)$, but this is a very tedious and error prone procedure. We will consider some tricks for entering such complex numbers more efficiently. For what follows, set the calculator numeric mode to standard to see the full 12 significant digits.

The easiest case is when *r* is "nice" and the argument is of the form $\theta = \frac{a}{b}\pi$, with *b* a divisor of 180. Such an argument is an integer when expressed in degrees, hence easy to enter in that form. For example, if $P = 2e^{\frac{\pi}{4}i}$, the argument is 45º. To get this complex number into the calculator in radian form, first set the calculator to degree mode, enter the number in degree form, then switch the calculator to radian mode; the 45 will change to .785398163398.

We now return to the more general case, $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$. Although there are several ways to handle this, we will discuss only one; the equation writer. (See page 2-10 and Appendix E of ***UG***.) With the calculator in radian mode, key in RS EQW 2 $\times$ $\sqrt{x}$ 3 RA $\times$ LS e^x LS π $\div$ 7 RA $\times$ LS i ENTER EVAL. You should now see $(3.46410161514,\angle .448798950513)$ on the display. Note that we have used the equation writer to create the algebraic object $2\sqrt{3}e^{\frac{\pi}{7}i}$, then used the evaluation command, EVAL, to convert it to the complex number (3.46410161514,∠.448798950513).

The concepts of negative, subtraction, and conjugate are all easily handled on the calculator. The +/- key converts the complex number in level 1 of the stack to its negative. Enter your favorite complex number and try it! Note that pressing +/- twice returns the original number as it should. The - key will subtract the complex number in level 1 from the complex number in level 2. Try it with your choice of complex numbers *A* and *B*. After computing *S* = *A* - *B*, verify that *S* + *B* returns your original *A*. There is also a conjugate function, but it is in one of the menus. Key in LS MTH NXT F3-CMPLX NXT and you will find CONJ as the third item in the menu. Key in a complex number and try it. Again, pressing it twice gives back the original complex number as expected.

There are two other functions on this menu page; NEG and SIGN. NEG works the same as the +/- key and SIGN converts the complex number on level 1 into a complex number with the same argument but with absolute value 1. Press NXT and you see six more functions related to complex numbers. The first four will be discussed in the next section, but ABS returns the magnitude of the complex number on level 1, and ARG returns its argument. ABS and ARG are also on the keyboard in conjunction with the ÷ key.

7. Rectangular Form on the Calculator

If the calculator is still in polar mode from the previous section, press MODE and change the Coordinate system back to Rectangular.   The complex number *a* + *b*i has rectangular form (*a*,b) on the calculator. For example, to enter 3 + 5i into the calculator key in LS ( ) 3 SPC (one could key in the comma here instead of SPC, but that would take two key strokes and the calculator will turn the space into a comma anyway) 5 ENTER. At this point we can also discuss the first four functions in the complex number menu which were mentioned in the previous section. Key LS MTH NXT F3-CMPLX to get back into the complex number menu. Assuming there is a complex number on level 1 of the stack (with the calculator in either rectangular or polar mode) F1-RE returns the real part of that complex number and F2-IM returns the imaginary part of the complex number on level 1. The function F3-C->R takes the complex number from level 1 then puts the real part on level 2 and the imaginary part on level 1 while F4-R->C reverses that process. It takes a real number from level 2 as the real part and a real number from level 1 as the imaginary part and forms a complex number which it puts on level 1.

Calculator Example 2.7.1

We will use the calculator to do Examples 1 and 2 from Section 5 of this chapter. For Example 1, we were to find the rectangular form of $P = 5\sqrt{2}e^{(3\pi/4)i}$. With the calculator set to rectangular mode and standard display, use the equation writer to key in the polar form of *P*: RS

EQW 5 × $\sqrt{x}$ 2 RA × LS e^x 3 × LS π UA ÷ 4 RA × LS i ENTER EVAL. The result will be (-4.99999999998,5). The real part should, of course, be -5, but we are seeing the unavoidable roundoff errors which occur when we use a finite device to approximate real numbers.

For Example 2 we were to find the polar form for $Q = -7 - 7\sqrt{3}i$. We will place the real part on level 2 of the stack and the imaginary part on level 1, then use the R->C command to create the complex number. Set the calculator to polar coordinates, standard display, radian angle mode, and bring up the complex number menu. Now key 7 +/- ENTER ENTER 3 $\sqrt{x}$ × F4-R->C. We should now see (14,∡−2.09439510239). There is one more thing we can try which sometimes gives us an answer in a form we would like to see it. Key F6-ARG LS CONVERT F4-REWRI NXT F6–→Qp and we see $-2/3*p$ on the display. The F6–→Qp command works to convert a decimal expression into a rational number times π if it's not to far from one of the "nice" angles. If, however, you compute $\frac{251\pi}{3163}$ as a decimal then try F6–→Qp to get it back, you get a very strange result.

Calculator Example 2.7.2

Find |(3 + 7i)(2 - 5i)|. We key in the problem: LS ( ) 3 SPC 7 ENTER LS ( ) 2 SPC 5 +/- × LS ABS. We see the answer 41.0121933088. Notice that it doesn't matter if the calculator is in rectangular or polar coordinates when we key in the problem.

Exercises for Chapter II Sections 6 and 7

For problems 1 - 8 give the magnitudes, real parts, and imaginary parts to two decimal places and the arguments in radians to four decimal places.

1. Let $C = 2e^{-3\pi i/4}$, $D = 4e^{\pi i/6}$, $E = e^{2\pi i}$, and $F = e^{-2\pi i/3}$. Give the polar form (i.e., $re^{\theta i}$ form) for each of the following products. Compare your answers to the exact values obtained in Problem 1 of Exercises for Chapter 2 Sections 1 and 2.

   (a) CD; (b) CE; (c) CF; (d) C^2.

2. Let $A = 1024e^{(\pi/4)i}$ and $C = e^{(2\pi/5)i}$.

   (a) Use the y^x key to find a $B = A^{1/5}$ in polar form.

   (b) Find in polar form CB, C^2B, C^3B, C^4B, and C^5B.

   (c) Compare your answers with the exact values obtained in Problem 19 of Exercises for

Chapter 2 Sections 1 and 2.

3. Find the polar form for 5 fifth roots of $249.47e^{0.7493i}$.

4. Find the polar form for 7 seventh roots of $8274.85e^{-0.4444i}$.

5. Convert the following polar forms to rectangular form $a + b$i.

(a) $8e^{(\pi/6)i}$; (b) $8e^{(5\pi/6)i}$; (c) $\sqrt{2}e^{(5\pi/4)i}$; (d) $9e^{(3\pi/2)i}$; (e) $4e^{(-\pi/4)i}$.

(f) Compare your answers with the exact values obtained in Problem 1 of Exercises for Chapter 2 Sections 3, 4, and 5.

6. Convert the following rectangular forms to polar form $re^{\theta i}$.

(a) 1 + i; (b) 7i; (c) $-5\sqrt{3} + 5i$; (d) -3; (e) -4 - 4i.

(f) Compare your answers with the exact values obtained in Problem 2 of Exercises for Chapter 2 Sections 3, 4, and 5.

7. Find complex numbers in polar and rectangular form with magnitude 1 and arguements equal to those listed below but converted to radians.

(a) 105^o; (b) 15^o; (c) 225^o; (d) 135^o; (e) 195^o.

(f) Compare your answers with the exact values obtained in Problem 16 of Exercises for Chapter 2 Sections 3, 4, and 5.

8. Find the rectangular form of $e^{(\pi/8)i}$ by entering $e^{(\pi/4)i}$ then pressing the square root key. Compare your answers with the exact values optained in Problem 32 of Exercises for Chapter 2 Sections 3, 4, and 5.

For problems 9 - 12 let $A = 3.47 - 6.52i$, $B = 4.32e^{1.1275i}$, $C = -8.46 + 3.92i$ and $D = 0.39e^{-2.0727i}$. Evaluate the given expressions and leave your answers in both polar and rectangular form. Give the magnitudes, real parts, and imaginary parts to two decimal places and the arguments in radians to four decimal places. HINT: It may be helpful to store the values of *A*, *B*, *C*, and *D* in your calculator memory.

9. (a) $(A + B - C)D$; (b) $\dfrac{A+B}{C-D}$; (c) $(A^2 - C^2)D$; (d) $(A + C)^3D$.

10. (a) $(A + B)(C - D)$; (b) $\dfrac{A - C}{B + D}$; (c) $(A^3 + C^3)D$ (d) $(A - C)^2D$.

11. Find the distance between A and B.

12. Find the distance between C and D.

In navigation directions are called bearings and are measured in degrees clockwise from north. Thus, if you are traveling east, you are traveling on a bearing of 90°; and if you are traveling south west, you are traveling on a bearing of 225°. For problems 13 and 14 give directions as bearings to the nearest degree and distances as miles to the nearest tenth of a mile.

13. A Coast Guard patrol boat leave the Coast Guard station and travels 13.1 miles on a bearing of 37°. It then turns to a bearing of 106° and travels 19.1 miles. From there it travels 13.2 miles on a bearing of 250° at which time it receives a distress call from a fishing boat. To reach the fishing boat it travels 11.4 miles on a bearing of 348°.

    (a) Covert all of the bearings above to angles measured counterclockwise from east.

    (b) There is an injured crewman on the fishing boat and the Coast Guard medic decides he should be helicoptered to a hospital. What distance and bearing should the helicopter fly from the Coast Guard station to the fishing boat? HINT: See Calculator Example 2.6.3.

    (c) It is known that the nearest hospital is 19.5 miles on a bearing of 166° from the Coast Guard station. What distance and bearing must the helicopter fly from the fishing boat to the hospital?

14. A geological team leaves their base camp in a jeep and travel across the desert for 3.1 miles on a bearing of 27°. From there they travel for 6.7 miles on a bearing of 257°. They leave that location on a bearing of 146°, but their jeep breaks down after 3.2 miles. They radio the base camp for a second jeep to come pick them up.

    (a) What direction and distance must the second jeep travel from the base camp to reach the stranded team?

    (b) It is known that the nearest repair facility is 6.4 miles on a bearing of 229° from the base camp. If the decision is made to tow the disabled jeep directly to the repair facility, what direction and distance should they travel from the site of the breakdown?

Chapter III

Trigonometry Using Complex Numbers

1. The Trigonometric Functions

Let θ be any real number. To obtain the trigonometric functions of θ one finds $r > 0$, a, and b such that $re^{\theta i} = a + bi$ and then uses the definitions:

$$\text{(D)} \qquad \begin{aligned} \cos(\theta) &= \frac{a}{r}, & \sin(\theta) &= \frac{b}{r}, & \tan(\theta) &= \frac{b}{a}, \\ \sec(\theta) &= \frac{r}{a}, & \csc(\theta) &= \frac{r}{b}, & \cot(\theta) &= \frac{a}{b}. \end{aligned}$$

The above three letter functions are abbreviations for cosine, sine, tangent, secant, cosecant, and cotangent, respectively. We note that the requirement $r > 0$ insures that cosine and sine are defined for all real numbers θ, but the other four functions will be undefined for values of θ which cause a zero in the denominator. [See Problem 10 below.] When any of these functions is defined, however, Problem 37, Exercises for Chapter 2 Sections 3, 4, and 5 shows that the definition is not ambiguous; that is, the functions are ***well defined***. For example, since $\sqrt{2}e^{(3\pi/4)i} = -1 + i$, we have

$$\cos(3\pi/4) = \frac{a}{r} = \frac{-1}{\sqrt{2}}, \ \sin(3\pi/4) = \frac{b}{r} = \frac{1}{\sqrt{2}}, \ \tan(3\pi/4) = \frac{b}{a} = \frac{1}{-1} = -1,$$

and the other three functions are the reciprocals of these. The same results would be obtained from $5\sqrt{2}e^{(3\pi/4)i} = -5 + 5i$ or any other nonzero complex number with $3\pi/4$ as argument.

The definitions show that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \ \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}, \ \sec(\theta) = \frac{1}{\cos(\theta)}, \text{ and } \csc(\theta) = \frac{1}{\sin(\theta)}.$$

The definitions also imply that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Then

$$re^{\theta i} = a + bi = r\cos(\theta) + r\sin(\theta)i = r\big(\cos(\theta) + i\sin(\theta)\big).$$

For complex numbers of absolute value 1, i.e., for $r = 1$, this becomes the ***Euler Formula***

$$e^{\theta i} = \cos(\theta) + i\sin(\theta).$$

Replacing θ with $-\theta$, we get $e^{-\theta i} = \cos(-\theta) + i\sin(-\theta)$. Taking conjugates of each side of the Euler Formula, we get $e^{-\theta i} = \cos(\theta) - i\sin(\theta)$. These equations show that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Example 1. ***Double angle formulas*** for cosine and sine.

We use the Euler Formula to obtain $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ as follows:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{2\theta i} = \left(e^{\theta i}\right)^2 \\ &= \left(\cos(\theta) + i\sin(\theta)\right)^2 \\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cdot\cos(\theta)\right).\end{aligned}$$

Equating the real and imaginary parts on each side of the equation, we have

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and

$$\sin(2\theta) = 2\sin(\theta)\cdot\cos(\theta).$$

These are the double angle formulas for cosine and sine respectively.

Similarly one can derive half angle formulas for $\cos(\theta/2)$ and $\sin(\theta/2)$ in terms of $\cos(\theta)$ using the fact that $e^{(\theta/2)i}$ is one of the two square roots of $e^{\theta i}$. [See Problems 8 and 18 below.]

Example 2. ***Addition Formulas*** for cosine and sine.

We use the Euler Formula to express $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ in terms of $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$ as follows:

$$e^{(\alpha + \beta)i} = e^{\alpha i}e^{\beta i} = \left(\cos(\alpha) + i\sin(\alpha)\right)\left(\cos(\beta) + i\sin(\beta)\right);$$

$$\begin{aligned}\cos(\alpha + \beta) + i\sin(\alpha + \beta) &= \left(\cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)\right) \\ &\quad + i\left(\sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta)\right).\end{aligned}$$

If we now equate the real and imaginary parts on each side of the equation, we get

$$\cos(\alpha + \beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)$$

and

$$\sin(\alpha + \beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta).$$

These are the addition formulas for the cosine and the sine, respectively. A symbolic aid for remembering these formulas is

$$(C + \mathrm{i}S)(c + \mathrm{i}s) = (Cc - Ss) + \mathrm{i}(Sc + Cs).$$

Example 3. ***Subtraction formula*** for tangent.

We seek $\tan(\alpha - \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$. First we note that

$$\begin{aligned} e^{\theta i}\sec(\theta) &= (\cos(\theta) + i\sin(\theta))\sec(\theta) \\ &= \cos(\theta)\cdot\sec(\theta) + i\sin(\theta)\cdot\sec(\theta) \\ &= \cos(\theta)\cdot\frac{1}{\cos(\theta)} + i\sin(\theta)\cdot\frac{1}{\cos(\theta)} \\ &= 1 + i\tan(\theta). \end{aligned}$$

What we need here is only that there is a real number k such that $ke^{\theta i} = 1 + i\tan(\theta)$. Taking conjugates, we have $ke^{-\theta i} = 1 - i\tan(\theta)$. Thus

$$\begin{aligned} re^{\alpha i}se^{-\beta i} &= (1 + i\tan(\alpha))(1 - i\tan(\beta)); \\ rse^{(\alpha-\beta)i} &= (1 + \tan(\alpha)\cdot\tan(\beta)) + i(\tan(\alpha) - \tan(\beta)). \end{aligned}$$

It now follows from the definition of the tangent that

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\cdot\tan(\beta)}.$$

This is the subtraction formula for the tangent.

Exercises for Chapter III Section 1

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. Given that $\tan(\theta) = 3/5$ and $\pi < \theta < 3\pi/2$:

   (a) Find *r, a, b* such that $re^{\theta i} = a + bi$ with the given angle θ.

   (b) Find the other 5 trigonometric functions of θ.

2. Given that $\cos(\phi) = 5/7$ and $-\pi < \phi < 0$, find a complex number in both polar and rectangular forms having ϕ as its argument and then find the other five trigonometric functions of ϕ.

3. Let β be an angle with $\sin(\beta) = \dfrac{2}{\sqrt{13}}$ and $\pi/2 < \beta < \pi$. Find *r, a, b* such that $re^{\beta i} = a + bi$ with the given β and then find $\cos(\beta)$ and $\tan(\beta)$.

4. Let $\tan(\alpha) = 8/15$ with α acute. Find a complex number in both polar and rectangular forms having α as its argument and then find $\cos(\alpha)$ and $\sin(\alpha)$.

5. Use the result of Problem 16 (b), Exercises for Chapter II Sections 3, 4, and 5 and definition (D) to find each of the following:

   (a) $\sin(15^o)$ (b) $\cos(15^o)$ (c) $\tan(15^o)$.

6. Use the result of Problem 16 (a), Exercises for Chapter II Sections 3, 4, and 5, the results of Example 3 in Section 5 of Chapter II and definition (D) to find each of the following:

   (a) $\sin(105^o)$; (b) $\cos(105^o)$; (c) $\tan(105^o)$;

   (d) $\sin(75^o)$; (e) $\cos(75^o)$; (f) $\tan(75^o)$.

7. Let α and β be as in Problems 3 and 4 above. Use **operations on complex numbers** and the results of Problems 3 and 4 to find the following: [Do **not** use subtraction formulas, double angle formulas, etc.]

   (a) $\cos(\alpha - \beta)$, (b) $\sin(2\beta)$, (c) $\tan\left(\dfrac{\pi}{2} - \beta\right)$, (d) $\cos(\pi - \alpha)$.

8. Given that $re^{\phi i} = -6 + 7i$, use square roots of complex numbers [See Problem 29, Exercises for Chapter II Sections 3, 4, and 5.] to find all possibilities for:

   (a) $\sin(\phi/2)$, (b) $\tan(\phi/2)$, (c) $\cos(\phi/2)$.

9. Use operations on complex numbers (not sum and difference formulas) to verify each of the following:

   (a) $\sin(\alpha + \dfrac{\pi}{2}) = \cos(\alpha)$; (b) $\cos(\alpha + \dfrac{\pi}{2}) = -\sin(\alpha)$;

   (c) $\sin(\alpha - \dfrac{\pi}{2}) = -\cos(\alpha)$; (d) $\cos(\alpha - \dfrac{\pi}{2}) = \sin(\alpha)$;

10. It is clear from definitions (D) that there are values of θ for which some of the trigonometric functions are not defined because the denominator of the defining fraction will be zero.

(a) Characterize all the values of θ for which the cosecant and cotangent are not defined.

(b) Characterize all the values of θ for which the tangent and secant are not defined.

11. Prove the ***Pythagorean Identity***; $\sin^2(\theta) + \cos^2(\theta) = 1$ for all angles θ.

12. Let $e^{\theta i} = c + is$. Find all six trigonometric functions of θ in terms of *c* and *s*.

13. Let $e^{\theta i} = c + is$. Find the six trigonometric functions of θ in terms of *c* for (a) $0 \le \theta \le \pi$ and (b) $\pi \le \theta \le 2\pi$. [HINT: Use the results of problems 11 and 12.]

14. Let $e^{\theta i} = c + is$. Express $e^{2\theta i}$, $e^{3\theta i}$, $e^{4\theta i}$, and $e^{5\theta i}$ in terms of *c* and *s*.

15. Use Problem 14 to express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ for $n = 1, 2, 3, 4, 5$.

16. Express $\cos(n\theta)$ and $\dfrac{\sin(n\theta)}{\sin(\theta)}$ in terms of $\cos(\theta)$ for $n = 1, 2, 3, 4, 5$.

17. Derive the addition, subtraction, and double angle formulas for the cosine, sine, tangent, and cotangent using complex numbers. (Some of these are in Examples 1, 2, and 3.)

18. (a) Derive the half angle formulas $\cos(\alpha/2) = \pm\sqrt{\dfrac{1 + \cos(\alpha)}{2}}$ and

$\sin(\alpha/2) = \pm\sqrt{\dfrac{1 - \cos(\alpha)}{2}}$ using square roots of complex numbers. Explain choice of $\pm$ sign. [See Problem 31, Exercises for Chapter II Sections 3, 4, and 5.]

(b) Use part (a) to show that $\cos^2(\beta) = \dfrac{1 + \cos(2\beta)}{2}$ and $\sin^2(\beta) = \dfrac{1 - \cos(2\beta)}{2}$.

(c) Use Problem 28, Exercises for Chapter II Sections 3, 4, and 5 to show that

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)}.$$

19. (a) Complete the following table: [HINT: See Problem 3, Exercises for Chapter II Sections 3, 4, and 5. and use $\sin(-\theta) = -\sin(\theta)$.]

x	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(x)$					0	$1/2$			

(b) Explain why $\sin(x + \pi) = -\sin(x)$ and $\sin(x + 2\pi) = \sin(x)$.

(c) Use parts (a) and (b) to tabulate $y = \sin(x)$ for $\pi/2 \le x \le 3\pi/2$.

(d) Graph $y = \sin(x)$ for $-\pi \le x \le 2\pi$.

20. (a) Graph $y = \cos(x)$ for $-\pi \le x \le 2\pi$.

(b) Graph $y = \tan(x)$ for $-\pi/2 < x < \pi/2$ and $\pi/2 < x < 3\pi/2$.

21. Use the formula $\csc(x) = 1/\sin(x)$ and Problem 19 to graph $y = \csc(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

22. Graph:

(a) $y = \sec(x)$ for $-\pi/2 < x < \pi/2$ and $-3\pi/2 < x < -\pi/2$.

(b) $y = \cot(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

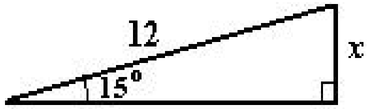
23. Consider a right triangle with α one of its acute angles. Let *hyp* be the length of the hypotenuse, *adj* the length of the side adjacent to α, and *opp* the length of the side opposite α. Verify each of the following:

(a) $\cos(\alpha) = \dfrac{adj}{hyp}$, (b) $\sin(\alpha) = \dfrac{opp}{hyp}$, (c) $\tan(\alpha) = \dfrac{opp}{adj}$,

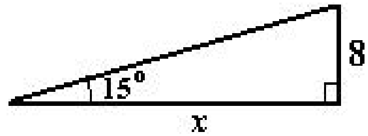
(d) $\sec(\alpha) = \dfrac{hyp}{adj}$, (e) $\csc(\alpha) = \dfrac{hyp}{opp}$, (f) $\cot(\alpha) = \dfrac{adj}{opp}$.

24. Use the results of problems 5 and 23 above to find x in each of the following.

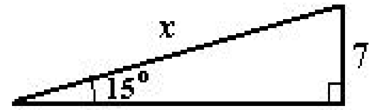
(a)



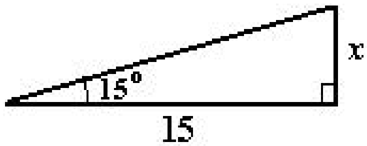
(b)



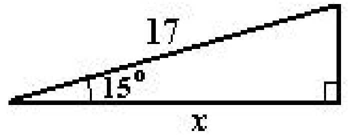
(c)



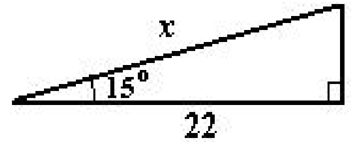
(d)



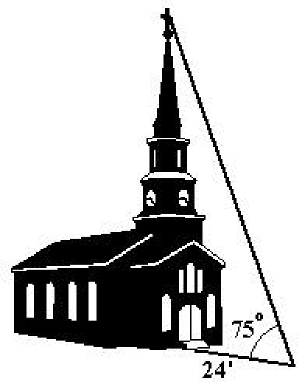
(e)



(f)



25. There is concern that a church steeple is too high for a new airport runway which is to be extended near by. An airport official stands at a point 24 feet from the point directly below the top of the steeple. From that location the angle of elevation to the top of the steeple is 75°. How tall is the steeple? [HINT: See problem 24 above.]



2. The Inverse Trigonometric Functions.

If f and g are functions such that f(a) = b if and only if g(b) = a, then f and g are ***inverse functions*** of each other. For example, the functions f and g with $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverse functions of one another since $b = a^3$ if and only if $a = \sqrt[3]{b}$. [Note that the inverse function of $f(x) = x^3$ is $g(x) = \sqrt[3]{x}$ while the additive inverse or negative is $-f(x) = -x^3$ and the multiplicative inverse or reciprocal is $1/f(x) = 1/x^3$.]

A function f that has the same value b for two different numbers a and c in its domain can not have an inverse function g since f(a) = b = f(c), with f and g inverses of each other, implies a = g(b) = c. Each of the trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant repeats its values in intervals of 2π and hence does not have an inverse function.

However the trigonometric functions with suitably restricted domains have inverses. For example, as x increases from $-\pi/2$ to $\pi/2$, $\sin(x)$ increases steadily from -1 to 1; hence the sine function with domain restricted to the interval $-\pi/2 \le x \le \pi/2$ does not repeat values and so has an inverse function. We use Sine (abbreviated Sin) to denote the sine function with domain the closed interval $[-\pi/2,\ \pi/2]$ and designate its inverse as Arcsine (Arcsin). The domain of the Arcsine function is [-1, 1] and its range is $[-\pi/2,\ \pi/2]$.

Similarly, the tangent on the open interval $-\pi/2 < x < \pi/2$ takes on all real values once and only once and so has an inverse. We designate the tangent function with domain restricted to the open interval $(-\pi/2,\ \pi/2)$ as Tangent (Tan) and its inverse as Arctangent (Arctan).

As x varies from 0 to π, $\cos(x)$ decreases steadily from 1 to -1. Therefore we use Cosine (Cos) to designate the restriction of the cosine function to the domain $[0,\pi]$; its inverse is written as Arccosine (Arccos).

The essential facts about these three inverse trigonometric functions are:

y = Arcsin(x) means that x = sin(y) and $-\pi/2 \le y \le \pi/2$,
y = Arctan(x) means that x = tan(y) and $-\pi/2 < y < \pi/2$,
y = Arccos(x) means that x = cos(y) and $0 \le y \le \pi$.

Frequently Arcsin(x), Arctan(x), and Arccos(x) are written $\sin^{-1}x$, $\tan^{-1}x$, and $\cos^{-1}x$ respectively. One should be prepared for this bad notation in the literature and not allow it to make one confuse an inverse function with a reciprocal. For example, $\sin^{-1}x$ is Arcsin(x) but is **not** $(\sin(x))^{-1} = \csc(x)$.

Exercises for Chapter III Section 2

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. What might be the motivation for choosing the interval $[0,\pi]$ as the domain of the Cosine function?

2. Give the domain and range of: (a) Arctan(x); (b) Arccos(x).

3. Find Arcsin(½) and four other real numbers x such that sin(x) = ½.

4. Find $\text{Arctan}\left(\sqrt{3}\right)$ and four other real numbers x such that $\tan(x) = \sqrt{3}$.

5. Find: (a) Arcsin(1); (b) Arctan(1); (c) Arccos(½).

6. Find: (a) $\text{Arcsin}\left(\frac{-\sqrt{2}}{2}\right)$; (b) $\text{Arctan}\left(-\sqrt{3}\right)$; (c) Arccos(-1).

7. (a) Graph both $y = \text{Cos}(x)$ and $y = \text{Arccos}(x)$ on the same axes.

(b) Are these graphs symmetric to each other with respect to some line?

8. Graph; (a) $y = \text{Arcsin}(x)$; (b) $y = \text{Arctan}(x)$.

3. Solving Triangles

There are three important congruence theorems from geometry that are usually abbreviated as ASA, SAS, and SSS. ASA, for example, tells us that if two angles and the included side of one triangle are equal to the respective two angles and included side of another triangle, the two triangles are congruent. Similar statements apply to the other two abbreviations. What these theorems tell us, is that if the right three parts of a triangle are known, the other three parts are fixed and they should be able to be found. Having partial information about a triangle and using it to find the rest of the information about the triangle is referred to as ***solving the triangle***. The procedure for solving right triangles is demonstrated in Problem 24 of Exercises for Chapter 3 Section 1. This sections introduces two important laws of trigonometry that can be used for solving any triangle for which appropriate information is known.

Law of Sines

Consider the triangles in Figure 1a and Figure 1b. Since every triangle has at least two acute angles, we can assume we have picked one of them to call α. Since $\sin(\alpha) = \frac{h}{b}$, (see Problem 23, Exercises for Chapter 3 Section 1.) $h = b\sin(\alpha)$. If β is acute as in Figure 1a, then $\sin(\beta) = \frac{h}{a}$. If β is obtuse as in Figure 1b, then $\beta' = 180^o - \beta$ and

$\sin(\beta) = \sin(180^o - \beta) = \sin(\beta') = \frac{h}{a}$. In either case, $h = a\sin(\beta)$. Equating these values of h, we have $b\sin(\alpha) = a\sin(\beta)$ or

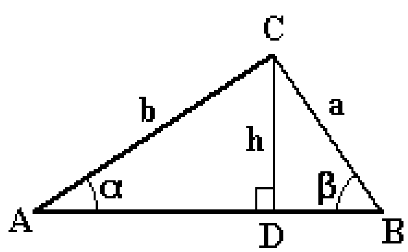


Figure 1a

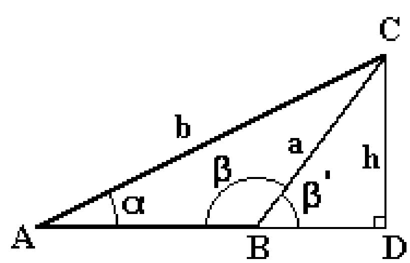


Figure 1b

$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b}$. Similarly, if γ is the angle at C and c is its opposite side, we can show that $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

We are now prepared to state the ***Law of Sines***: If a triangle has angles α, β, and γ with opposite sides a, b, and c respectively, (See Figure 1c) then

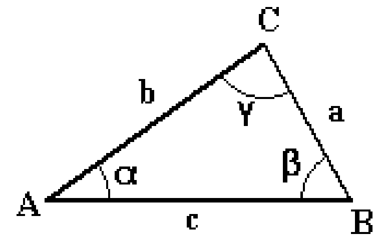


Figure 1c

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

All of the examples and problems which follow are based on triangles labeled as in Figure 1c.

Example 1. If $\alpha = 45^\circ, \beta = 60^\circ$, and $c = 8$, find angle C and sides a and b. (Given ASA.)

Solution: First, $\gamma = 180^\circ - (45^\circ + 60^\circ) = 75^\circ$. From Problems 6 and 19 of Exercises for Chapter III Section 1 we have $\sin(\alpha) = \frac{1}{2}\sqrt{2}$, $\sin(\beta) = \frac{1}{2}\sqrt{3}$, and $\sin(\gamma) = \frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)$. Now, from the Law of Sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$, thus $a = \frac{c\sin(\alpha)}{\sin(\gamma)} = \frac{8\left(\frac{1}{2}\sqrt{2}\right)}{\frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)} = 8\left(\sqrt{3}-1\right)$.

Similarly, $b = 4\sqrt{6}\left(\sqrt{3}-1\right)$.

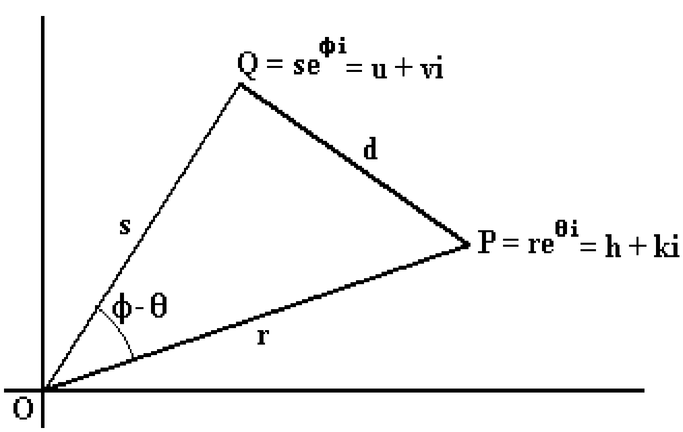


Figure 2

Law of Cosines

Consider the complex numbers P and Q as shown in Figure 2. By Problem 33 of Exercises for Chapter II Sections 3, 4, and 5, we see that d, the distance between P and Q, is

$$d = |Q - P| = \sqrt{(u-h)^2 + (v-k)^2}.$$

But $h = r\cos(\theta)$, $k = r\sin(\theta)$, $u = s\cos(\phi)$, $v = s\sin(\phi)$. Substituting these into the equation for d, squaring both sides, expanding, and collecting like terms, gives us

$$
\begin{aligned}
d^2 &= (s\cos(\phi) - r\cos(\theta))^2 + (s\sin(\phi) - r\sin(\theta))^2 \\
&= s^2(\cos^2(\phi) + \sin^2(\phi)) + r^2(\cos^2(\theta) + \sin^2(\theta)) - 2sr(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)).
\end{aligned}
$$

Using the Pythagorean Identity [See Problem 11, Exercise for Chapter III Section 1.] and the subtraction formula for the cosine [See Problem 17, Exercise for Chapter III Section 1.] this becomes

$$d^2 = s^2 + r^2 - 2sr\cos(\phi - \theta).$$

If ΔOPQ is now relabeled as in Figure 3c, we get the three forms of the ***Law of Cosines***:

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(\alpha) \\
b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\
c^2 &= a^2 + b^2 - 2ab\cos(\gamma)
\end{aligned}
$$

depending on whether angle A, B or C, respectively is placed at the origin.

Example 2. If $\gamma = 15^\circ$, $a = 5\sqrt{6}$, and $b = 10$, find side c and angles A, and B. (Given SAS.)

Solution: From Problem 5 of Exercises for Chapter 3 Section 1, $\cos(15^\circ) = \frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)$.

Then, from the Law of Cosines,

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos(\gamma) = 150 + 100 - 50\sqrt{3}\left(1+\sqrt{3}\right) \\
&= 100 - 50\sqrt{3} = 25\left(4 - 2\sqrt{3}\right) = 5^2\left(\sqrt{3}-1\right)^2.
\end{aligned}
$$

Thus, $c = 5\left(\sqrt{3}-1\right)$. From the Law of Sines $\frac{\sin(\beta)}{b} = \frac{\sin(15^\circ)}{c}$. Since

$\sin(15^\circ) = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$,

$$
\begin{aligned}
\beta &= \operatorname{Arcsin}\left(\frac{b\sin(\gamma)}{c}\right) = \operatorname{Arcsin}\left(\frac{10\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)}{5\left(\sqrt{3}-1\right)}\right) \\
&= \operatorname{Arcsin}\left(\frac{1}{2}\sqrt{2}\right) = 45^\circ.
\end{aligned}
$$

Finally, $\alpha = 180^\circ - (15^\circ + 45^\circ) = 120^\circ$.

Exercises for Chapter III Section 3

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. (a) Reconsider Examples 2. After finding c with the Law of Cosines, use the Law of Sines and Arcsin to find α, then use subtraction to find β.

   (b) Why are the answers different from the correct answers given in Example 2?

   (c) What must be done to insure getting the correct answers in this type of problems?

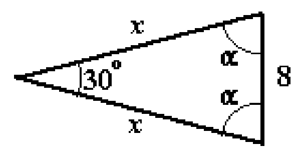
2. Solve each of the following triangles (See Figure 1c) from the given information if possible. The results of Problems 5 and 6 of Exercises for Chapter III Section 1 may be helpful here.

   (a) $\alpha = 45°, \beta = 30°$, and $c = 12$.

   (b) $\beta = 15°, a = 10\sqrt{2}\left(3 - \sqrt{3}\right)$, and $c = 20$. HINT: $a < c$ in this case.

   (c) $a = 3\left(\sqrt{3} - 1\right), b = 3\sqrt{2}$, and $c = 6$. HINT: Use the Law of Cosines first to find one of the angles.

3. Solve the isosceles triangle in the figure to the right.



4. Show that if you are given two angles and a side which is not the included side, (AAS or SAA), the triangle is still determined. This is usually a corollary to the ASA theorem.

5. Why is AAA not a congruence theorem?

6. Show that SSA does not determine a triangle by finding two distinct triangles which satisfy $\alpha = 30°, a = 5\sqrt{2}\left(\sqrt{3} - 1\right)$, and $b = 10$.

4. Trigonometry on the Calculator.

We note that the calculator has keys labeled SIN, COS, and TAN. These are, as might be expected, the keys for the sine, the cosine, and the tangent functions respectively. If the real number x is on level 1 of the stack, pressing SIN will give sin(x), TAN will give tan(x), and COS will give cos(x). The real number x will be interpreted as degrees or radians according to the angle mode setting. For example if the number 30 is on level 1 of the stack and the calculator is in degree mode, pressing SIN gives the result .5 as expected. If, however, the calculator is in radian mode with 30 on level 1 of the stack, the result of pressing SIN is -.988031624093, which is the sine of 30 radians.

There are no keys for the cosecant, secant, or cotangent functions. Since these functions are the reciprocals of the sine, cosine, and tangent respectively, they can be obtained with the keys we have and the 1/x key. The trigonometric functions are on pages 3 - 6 of ***UG***.

Calculator Example 3.4.1

Find $\sec(\pi/3)$.

Solution: With the calculator in radian mode key in LS π 3 $\div$ COS 1/x. We see the result 2.00000000001. The answer, of course, should be 2, but we are again seeing an example of the round off errors caused by using a finite machine to approximate computations with real numbers.

We see that the left shift functions for SIN, COS, and TAN are labeled ASIN, ACOS, and ATAN, respectively. These are, respectively, the Arcsine, Arccosine, and Arctangent functions. These functions are not inverses of each other since, for example, SIN is the sine function, not the Sine function.

Calculator Example 3.4.2

Find $\tan(3\pi/4)$; then take the Arctangent of the result.

Solution: With the calculator in radian mode key in 3 LS π $\times$ 4 $\div$ TAN, and we see the expected answer -1. Now key in LS ATAN and we get the result -.785398163397, which is the decimal approximation for $-\pi/4$. This should not be surprising since $3\pi/4$ is not in the domain of Tan and so $\text{Arctan}(\tan(\theta))$ is not necessarily θ for this angle.

Before starting the next example, you may want to review the plotting instruction in Chapter 12 of ***UG***.

Calculator Example 3.4.3

Graph $y = \sin(x)$ for $-180° \le x \le 360°$ on the calculator.

Solution: The first step is to insure your calculator will react as indicated by these instructions. If you have variables called X, EQ, and/or PPAR in your variable list, purge them. (See page 2-32 and 2-33 of ***UG***.)

Set the calculator to degree mode. Now key in LS(hold) 2D/3D to get into the PLOT SETUP dialog box. (**REMEMBER** - If your calculator is in RPN mode, you must **HOLD** LS while pressing 2D/3D or WIN to get into those dialog boxes.) Press DA ' SIN F1-X ENTER to enter 'SIN(X)' into the EQ: box. We now want to set the horizontal ticks to 30 degrees and the vertical ticks to .2 units. Press DA 30 ENTER .2 ENTER F2-CHK. Your screen should now look like Figure 3a.

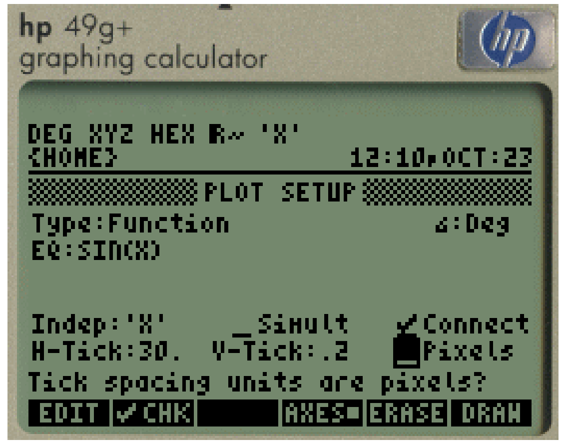


Figure 3a

Now key in NXT F6-OK LS(hold) WIN to get into the PLOT WINDOW dialog box. Now key 180 +/- ENTER 360 ENTER to set the H-VIEW: boxes and F4-AUTO to have the calculator select the best values for V-VIEW: boxes. The dialog box should now look like Figure 3b.

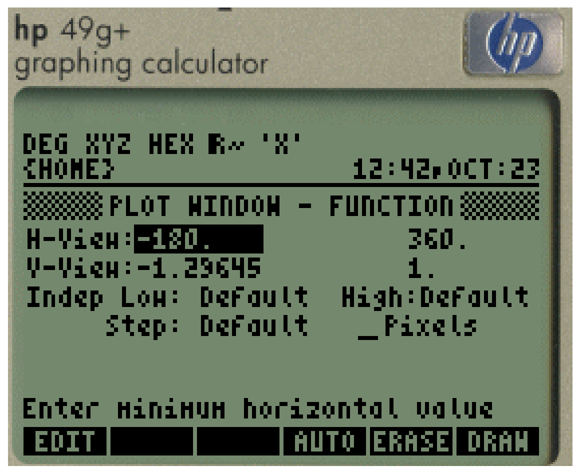


Figure 3b

Now press F5-ERASE F6-DRAW to draw the graph which we see in Figure 3c.

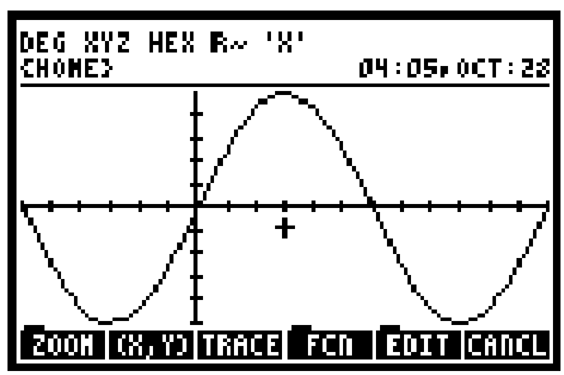


Figure 3c

Calculator Example 3.4.4

Graph *y* = Cos(*x*) and *y* = Arccos(*x*) on the same axes. See Problem 7, Exercises for Chapter III Section 2.

Solution: Assuming your calculator is as you left it from the previous example, press CANCEL or F6-CANCL to return to the PLOT WINDOW dialog box. Now key NXT F1-RESET DA F6-OK. This resets the plot parameters back to their default state. In this state one unit in both directions is 10 pixels, so geometric properties will not be distorted, but the two views are about twice as big as we need them. We will cut each of the fields in the two view areas in half, and will add .5 to the two V-VIEW fields to move the origin down that amount. To accomplish this, press F2-CALC 2 ÷ F6-OK to set the left field of H-VIEW. Now key NXT F2-CALC 2 ÷ F6-OK to set the right field of H-VIEW. For each of the two fields of V-VIEW key NXT F2-CALC 2 ÷ .5 + F6-OK.

Now press 0 ENTER so the graph will start at $x = 0$ and LS π to end the graph at $x = \pi$. Your screen should now look like Figure 4a.

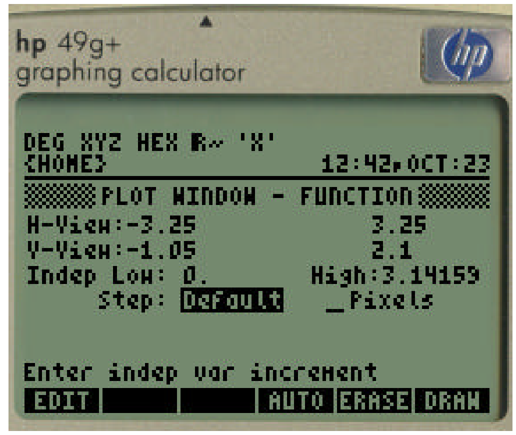


Figure 4a

Now press NXT F6-OK LS(hold) 2D/3D to get into the PLOT SETUP dialog box. Key RA +/- to change the angle mode to radians, then RA ‹ COS F1-X ENTER to change to the cosine function. Now key DA .5 ENTER .5 ENTER to change the ticks to .5 units in both directions. The screen should now look like Figure 4b.

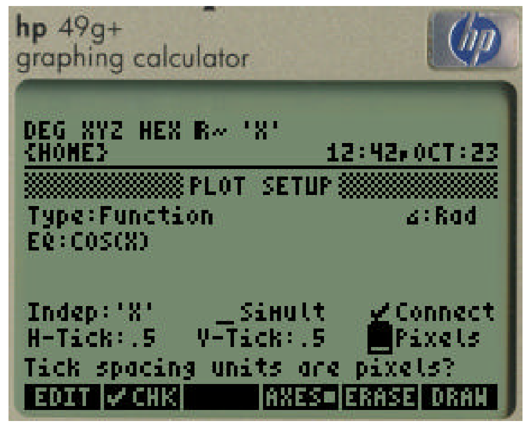


Figure 4b

Now press F5-ERASE F6-DRAW, and when the graph is complete, press the minus sign to remove the menu. You should see the graph shown in Figure 4c.

Press CANCEL to get back to the PLOT SETUP dialog box and change the EQ: field to ACOS(X). Now press LS(hold) WIN to get into the PLOT WINDOW and change Indep Low: to -1 and High: to 1 to plot the domain [-1, 1]. Press F6-DRAW **without** erasing. You should now see the graph shown in Figure 4d.

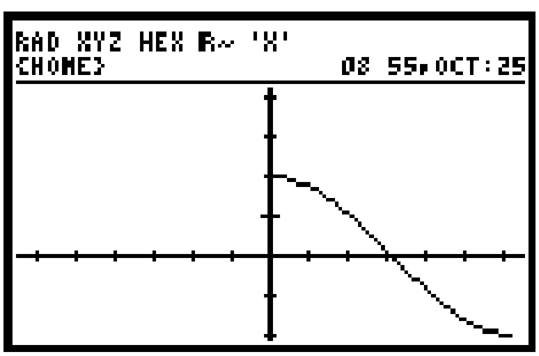


Figure 4c

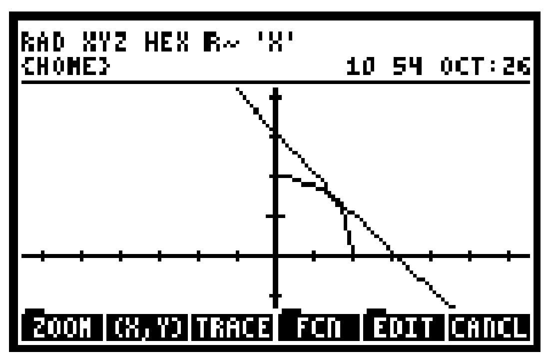


Figure 4d

Finally, let us add the line of symmetry for $\cos(x)$ and $\text{Arccos}(x)$. Press F2-(X,Y) and use RA and UA to move the cursor until the coordinates at the bottom of the screen show X:2. Y:2. (NOTE: holding the arrow keys down will cause the cursor to move more quickly.) Now press × to put a mark on the screen as shown in Figure 4e. Next press + to get the menu back, then press F5-EDIT. Use DA and LA to move the cursor back to the origin, press F3-LINE, then $\times$, and you should see the graph shown in Figure 4f.

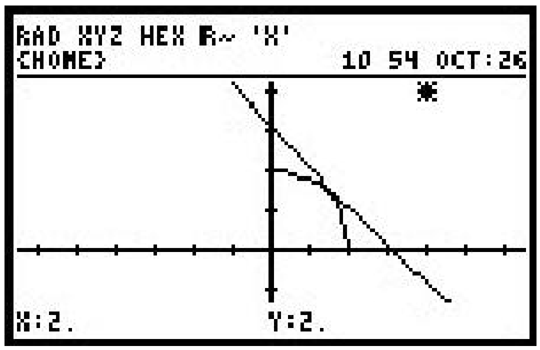


Figure 4e

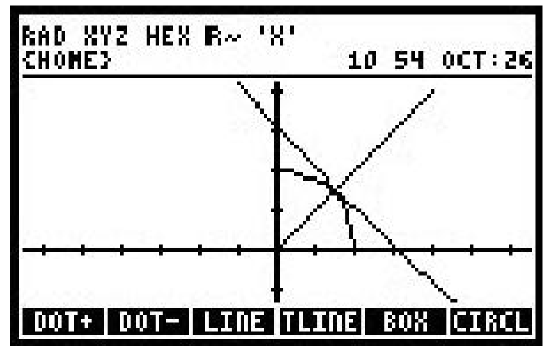


Figure 4f

Calculator Example 3.4.5

In ΔABC, $\alpha = 37.1^o$, $\beta = 58.3^o$, and $c = 47.26$. Solve the triangle. Find the angles to one decimal place and the lengths to two decimal places.

Solution: First $\gamma = 180^o - \alpha - \beta$, so on the calculator we key 180 ENTER 37.1 - 58.3 - and we see that $\gamma = 84.6°$. From the Law of Sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$. We solve this for a and substitute the known quantities to get $a = \frac{47.26\sin(37.1^o)}{\sin(84.6^o)}$. With the calculator in degree mode and set to Fix 2, we key in 47.26 ENTER 37.1 SIN $\times$ 84.6 SIN $\div$ and obtain $a = 28.63$.
Similarly

$$b = \frac{c\sin(\beta)}{\sin(\gamma)} = \frac{47.26\sin(58.3°)}{\sin(84.6°)} = 40.39.$$

Calculator Example 3.4.6

Given that in ΔABC one has $\gamma = \pi/6$, $a = 22.16$, and $b = 43.26$, solve the triangle. Give angles to three decimal places and lengths to 2 decimal places.

Solution: We observe that the given information is of the form SAS, hence the triangle is determined. We first use the Law of Cosines to find

$$c = \sqrt{a^2 + b^2 - 2ab\cos(\gamma)}$$

$$= \sqrt{22.16^2 + 43.26^2 - 2 \cdot 22.16 \cdot 43.26\cos(\pi/6)}.$$

On the calculator (in radian mode) this is accomplished with 22.16 LS x^2 43.26 LS x^2 + 2 ENTER 22.16 × 43.26 × LS π 6 ÷ COS × − $\sqrt{x}$, which gives us $c = 26.50$. We now use the Law of Sines to find

$$\sin(\alpha) = \frac{a\sin(\gamma)}{c}$$

or

$$\alpha = \operatorname{Arcsin}\left(\frac{22.16\sin(\pi/6)}{26.50}\right).$$

Before we continue, a reminder. The SWAP command can be found by TOOL F3-STACK F2-SWAP. It can also be found by LS RA, and if the calculator is in a state where RA would not make sense, the LS is not necessary.

Set the calculator to Fix 3. Now, assuming the value of c is still on the stack from the previous calculation, we proceed with 22.16 ENTER LS π 6 ÷ SIN × SWAP ÷ LS ASIN, which gives us $\alpha = .431$. (Notice that when we were ready to divide by c we took advantage of the fact that it was already on the stack and just used the SWAP command to put it in the right position for the division. Every time one can eliminate the need to key in a number, one has removed an opportunity to make an error.) Finally, we compute the last angle by subtracting the first two from π and get $\beta = 2.187$.

Exercises for Chapter III Section 4

In all of the following problems give trigonometric functions to 4 decimal places, find all angles in the same units as angles given in the problem with 1 decimal place for degrees and 3 decimal places for radians, and give all lengths with the same number of decimal places as lengths given in the problem.

1. Find the six trigonometric functions for $\alpha = 38.4°$.

2. Find the six trigonometric functions for $\beta = 7\pi/12$.

3. Find the six trigonometric functions for $\gamma = -0.447$ radians.

4. Give each of the following in degrees:

   (a) Arcsin(0.7739); (b) Arcsin(-0.7739); (c) Arccos(0.7739); (d) Arccos(-0.7739);

(e) Arctan(0.7739); (f) Arctan(-0.7739).

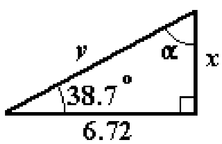
5. Find an angle in radians whose secant is 2.4483.

6. Graph cos(θ) for $-\pi/2 \le \theta \le 3\pi/2$ and cos(θ − π/6) for $-\pi/3 \le \theta \le 5\pi/3$ on the same plot on the calculator.

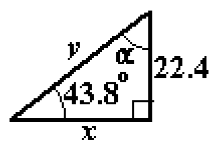
7. Do Problem 24 of Exercises for Chapter III Section 1 on the calculator and compare your answers to the exact answers.

8. Solve each of the following right triangles:

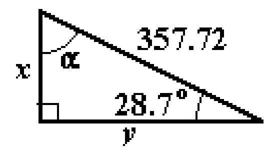
(a)



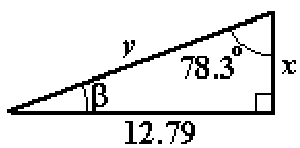
(b)



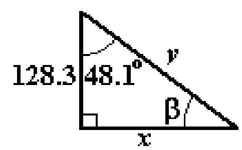
(c)



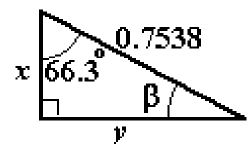
(d)



(e)

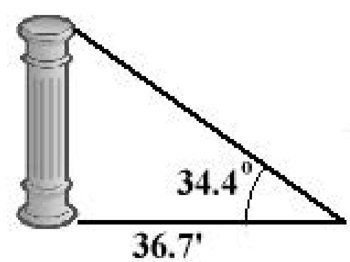


(f)



9. Do Problem 25 of Exercises for Chapter III Section 1 on the calculator and compare your answers to the exact answers.

10. An architect is interested in finding the height of the column shown on the right. She goes to a point 36.7 feet from the base of the column, from which point the angle of elevation to the top is 34.4°. How high is the column?



11. Use the calculator to solve Problem 2 of Exercises for Chapter III Section 3 and compare your answers to the exact solutions.

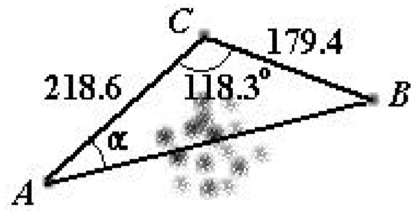
12. Solve each of the following triangles (See Figure 1c) from the given information if possible.

(a) $\alpha = 72.3°$, $\beta = 47.6°$, $b = 39.47$;

(b) $\beta = 5\pi/12$, $a = 2.917$, $c = 3.264$;

(c) $a = 472.6$, $b = 515.1$, $c = 497.7$, find angles in radians.

13. A contractor needs to run a sewer line from a new building at *A* to the waist treatment plant at *B*, but heavy underbrush makes it difficult to see directly from *A* to *B*. From *C*, however, he can clearly see both *A* and *B*. The distance *AC* is 218.6 meters, the distance *BC* is 179.4 meters, and the angle at *C* is 118.3 degrees. Find the distance *AB* and the angle α at *A*.



Supplementary Problems

These problems are intended for more advanced students. Some of them require concepts from Calculus and Linear Algebra.

1. Use the identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ to solve each of the following equations:

   (a) $4x^3 - 3x = \frac{1}{2}$; (b) $8x^3 - 6x - \sqrt{3} = 0$; (c) $x^3 - 3x + 1 = 0$.

2. Convert each of the following functions into the form $r\sin(\theta - \alpha)$, with appropriate constants r and α, and thus find the maximum and minimum values of $f(\theta)$ without using differentiation. Check using calculus.

   (a) $f(\theta) = 2\sin(\theta) - 2\sqrt{3}\cos(\theta)$;

   (b) $f(\theta) = 20\cos(\theta) + 21\sin(\theta)$.

3. Prove the following identities and generalize:

   (a) $\dfrac{\sin(\theta)}{\sin\left(\frac{\theta}{8}\right)} = 8\cos\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{4}\right)\cos\left(\dfrac{\theta}{8}\right)$;

   (b) $\dfrac{\sin(\theta)}{\sin\left(\frac{\theta}{9}\right)} = \left[4\cos^2\left(\dfrac{\theta}{3}\right) - 1\right]\cdot\left[4\cos^2\left(\dfrac{\theta}{9}\right) - 1\right]$.

4. Let $P = e^{\theta i} = \cos(\theta) + i\sin(\theta)$. Show the following:

   (a) $P^n = \cos(n\theta) + i\sin(n\theta)$;

   (b) $1 + P + P^2 + \cdots + P^{n-1} = \dfrac{1 - P^n}{1 - P}$;

   (c) $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos[(n-1)\theta] =$

$$\frac{[1 - \cos(\theta)][1 - \cos(n\theta)] + \sin(\theta)\sin(n\theta)}{2 - 2\cos(\theta)};$$

(d) $\dfrac{\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta)}{\cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta)} = \tan\left[\dfrac{(n+1)\theta}{2}\right]$.

5. Use the Euler Formula $e^{\theta i} = \cos(\theta) + i\sin(\theta)$ to express

   (a) $\cos(\theta)$ in terms of $e^{\theta i}$ and $e^{-\theta i}$;

   (b) $\sin(\theta)$ in terms of $e^{\theta i}$ and $e^{-\theta i}$.

6. This problem assumes familiarity with Calculus.

   (a) Find the Taylor series about 0 of cos(x) and sin(x). Note: It is known that both of these series still converge if the real number x is replaced by any complex number z.

   (b) Use the results of part (a) and the Euler Formula to find a series for e^{xi}.

   (c) Replace x with -i in the result of part (b) and simplify the resulting series. How does the "e" introduced in this text compare to the "e" (that is, the base of the natural logarithm) introduced in Calculus?

7. Prove that $2\sin\left(\dfrac{\alpha - \beta}{2} + \dfrac{\pi}{4}\right)\cos\left(\dfrac{\alpha + \beta}{2} + \dfrac{\pi}{4}\right) = \sin(\beta) + \cos(\alpha)$.

8. Show that $\sin(x + y + z) = \sin(x)\cos(y)\cos(z) + \cos(x)\sin(y)\cos(z)$
   $+ \cos(x)\cos(y)\sin(z) - \sin(x)\sin(y)\sin(z)$.

9. Show that $\cos(x + y + z) = \cos(x)\cos(y)\cos(z) - \sin(x)\sin(y)\cos(z)$
   $- \sin(x)\cos(y)\sin(z) - \cos(x)\sin(y)\sin(z)$.

10. (a) With the calculator in radian mode put 'SIN(X)' on level 2 of the stack and, 'X' on level 1, then press RS ∂.

    (b) Leave the result of (a) on the stack but change the mode to degrees and repeat part (a).

    (c) Why do you get different results to parts (a) and (b)?

Problems 11 and 12 assume familiarity with Linear Algebra.

11. Let the linear operator $T(\mathbf{v})$ be defined by $T(\mathbf{v}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}\mathbf{v},$ let ΔOAB be as in Problem 42 of Exercises for Chapter 2 Sections 3, 4, and 5, let $\mathbf{v} = \overrightarrow{OA},$ and let $\mathbf{w} = \overrightarrow{OB}.$ Verify that $T(\mathbf{v})$ and $T(\mathbf{w})$ have the same effect on ΔOAB as multiplying A and B by $e^{(2/3)\pi i}$ had on that triangle. In this sense, the linear operator T and multiplying by $e^{(2/3)\pi i}$ are equivalent operations.

12. Show that the linear operator $T(\mathbf{v}) = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\mathbf{v}$ and multiplying by $re^{\theta i}$, $r > 0$, are equivalent processes in the sense given in Problem 11 above.

13. Sketch and identify the locus of all points A in the Argand Plane that satisfy the following equations:

(a) $|A - 3i| + |A + 3i| = 10.$

(b) $|A - 3i| - |A + 3i| = 1.$

Answers and Hints for Selected Odd Numbered Problems

Exercises for Chapter I

1. acute; $\sqrt{37}°$; right; $90°$: obtuse; $156°$: The rest are not possible in a triangle.

3. (i) (a) $3\sqrt{2}$; (b) $\frac{11}{2}\sqrt{3}$; (c) 8 (d) $\sqrt{33}$.
   (ii) (a) 4.24 (b) 9.53 (c) 8.00 (d) 5.74

5. $x = 9/2$ $y = 6$.

7. (a) Yes (b) No

9. (a) $u = \frac{a}{r^2}$; $v = \frac{b}{r^2}$.
   (b) $r = 2.823$; $u = 0.326$; $v = 0.138$

13. Hint: Make repeated use of the theorem in Section 5.

15. Hint: Use Problem 14.

17. (a) 13 (b) 15 (c) $\frac{\pi}{6}$; (d) $\frac{\pi}{4}$.

Exercises for Chapter II Sections 1 and 2

1. (a) $8e^{-7\pi i/12}$; (b) $2e^{5\pi i/4}$; $2e^{-17\pi i/12}$; (d) $4e^{-3\pi i/2}$.

3. (a) $6e^{11\pi i/12}$.

5. $|G + H| = 5$.

9 (a) The line *OU*. (b) The line through *O* perpendicular to *OU*.

11. $6e^{13\pi i/9}$.

15. (a) $16e^{\pi i}$; (b) $28e^{2\pi i}$; (c) $28e^{\pi i}$; (d) $28e^{\pi i}$.

17. (a) Yes; (b) Yes; (c) Yes.

19 (b) $4e^{\pi i/20}$.

Exercises for Chapter II Sections 3, 4, and 5

1. (a) $4\sqrt{3} + 4i$; (b) $-4\sqrt{3} + 4i$; (c) -1 - i;

   (d) -9i; (e) $2\sqrt{2} - 2\sqrt{2}i$.

5. -7 - 8i.

7. (a) $13e^{-\Phi i} = -12 - 5i$; (b) $13e^{(\pi-\Phi)i} = 12 + 5i$;

   (c) $13e^{(\Phi+\pi)i} = 12 - 5i$; (d) $13e^{\left(\frac{\pi}{2}-\Phi\right)i} = 5 - 12i$.

9. (a) $5e^{(\alpha+\pi)i} = -3 - 4i$; (b) $6\sqrt{2}e^{(7\pi/4)i} = 6 - 6i$;

   (c) $5e^{-\alpha i} = 3 - 4i$; (d) $6\sqrt{2}e^{-(3\pi/4)i} = -6 - 6i$;

   (e) $30\sqrt{2}e^{(\alpha+3\pi/4)i} = -42 - 6i$; (f) $30\sqrt{2}e^{(\alpha-3\pi/4)i} = 6 - 42i$.

11. (a) 56 + 33i; (b) 16 + 63i.

13. $\sqrt{74}$.

15. $\sqrt{185}$.

19. (a) $5e^{(\alpha+\pi/2)i} = 4 + 3i$; (b) $5e^{(\alpha+\pi)i} = -3 + 4i$;

    (c) $5e^{(\alpha-\pi/2)i} = -4 - 3i$; (d) $5e^{-\alpha i} = 3 + 4i$;

    (e) $5e^{\left(\frac{\pi}{2}-\alpha\right)i} = -4 + 3i$; (f) $25e^{2\alpha i} = -7 - 24i$;

    (g) $125e^{3\pi i} = -117 - 44i$.

21. $P = 10e^{-(\pi/3)i} = 5 - 5\sqrt{3}i;$ $\overline{P} = 10e^{(\pi/3)i} = 5 + 5\sqrt{3}i;$

$-P = 10e^{(2\pi/3)i} = -5 + 5\sqrt{3}i;$ $-\overline{P} = 10e^{-(2\pi/3)i} = -5 - 5\sqrt{3}i.$

23. (a) 12 + 5i; (b) 33 + 56i; (c) -12 - 5i; (d) 7 + 24i; (e) 63 + 16i.

25. $-134217728 + 134217728\sqrt{3}i.$

27. (c) $2\sqrt{117}.$

Exercises for Chapter II Sections 6 and 7

3. $3.02e^{0.1499i}$ $3.02e^{1.4065i}$ $3.02e^{2.6631i}$

$3.02e^{-2.3634i}$ $3.02e^{-1.1068}$

9. (a) $5.95e^{-2.5156i} = -4.82 - 3.49i;$ (b) $0.64e^{-3.1229i} = -0.64 - 0.01i;$

(c) $34.79e^{0.8303i} = 23.47 + 25.68i;$ (d) $69.48e^{2.5099i} = -56.07 + 41.03i.$

11. 10.55

13. (a) 53°; 344°; 200°; 102°.

(b) 16.5 miles on a bearing of 44°.

(c) 31.5 miles on a bearing of 192°.

Exercises for Chapter III Section 1

1. (a) $r = \sqrt{34};$ $a = -5;$ $b = -3.$

(b) $\cos(\theta) = -\frac{5}{\sqrt{34}}$; $\sin(\theta) = -\frac{3}{\sqrt{34}}$; $\cot(\theta) = \frac{5}{3}$

$\sec(\theta) = -\frac{\sqrt{34}}{5}$; $\csc(\theta) = -\frac{\sqrt{34}}{3}$.

3. $r = \sqrt{13}$; $a = -3$; $b = 2$; $\cos(\beta) = -\frac{3}{\sqrt{13}}$; $\tan(\beta) = -\frac{2}{3}$.

5. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$; (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$; (c) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

7. (a) $-\frac{29}{17\sqrt{13}}$; (b) -12/13; (c) -3/2; (d) -15/17

11. HINT: Make use of Euler formula and its conjugate.

Exercises for Chapter III Section 2

3. $\text{Arcsin}(1/2) = \frac{\pi}{6}$; $\frac{\pi}{6} + 2\pi n$ and $\frac{5\pi}{6} + 2\pi n$ for any integer *n*.

5. (a) $\frac{\pi}{2}$; (b) $\frac{\pi}{4}$; (c) $\frac{\pi}{3}$.

Exercises for Chapter III Section 3

3. $\alpha = 75°$; $x = \frac{8\left(\sqrt{3}+1\right)}{\sqrt{2}}$.

Exercises for Chapter III Section 4

1. $\cos(38.4^o) = 0.7837$; $\sin(38.4^o) = 0.6211$; $\tan(38.4^o) = 0.7926$;

$\sec(38.4^o) = 1.2760$; $\csc(38.4^o) = 1.6099$; $\cot(38.4^o) = 1.2617$.

3. cos(-0.447) = 0.9017; sin(-0.447) = -0.4323; tan(-0.447) = -0.4794;

   sec(-0.447) = 1.1090; scs(-0.447) = -2.3134; cot(-0.447) = -2.0861.

5. 1.150

13. Distance is 342.3 m and the angle is 27.5^o.

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